StickyBot Model

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1 StickyBot Model

This model depicts StickyBot in the dorsal plane, parallel to StickyBot's climbing surface. Each contact point is treated as a rotational joint. Each leg is modeled as a serial mechanism and is connected to the StickyBot body at a fully compliant hip, which ensures that the body of StickyBot retains full mobility. (3 DOF in the planar case, 6 for spatial).

The current method relies upon simple limbs consisting of one segment connecting the foot (rotational joint) to the hip (massless compliant prismatic-prismatic-revolute) joint. No damping is currently included, so, at best, a marginally stable system is expected. Adding damping is easily done; it just complicates the equations.

At a given instant, the limb joint angles and the body position/orientation are known. From the differences in the body position and the end of each limb, the forces on the body and the end of the limbs are known. The sum of forces and moments defines the resulting acceleration of the body's center of mass. The limbs' accelerations are then determined by the inverse dynamics of each limb.

Some notation:

$\mathbf{x}_{hb_i}$	Location of the i-th hip joint on the body
$\mathbf{x}_{hl_i}$	Location of the end of the i-th limb
$\mathbf{x}_{f_i}$	Location of the i-th foot
$\mathbf{x}_{lcg_i}$	Location of the i-th limb's center of mass
$\mathbf{x}_{cg}$	Location and orientation of the robot body
$\mathbf{r}_i$	Location of i-th hip with respect to the CG (fixed)
$\mathbf{r}_{hl_i}$	Location of i-th hip with respect to the i-th foot
$\mathbf{K}_i$	Hip stiffness for the i-th hip
${}^g\mathbf{R}^b$	Rotation matrix from body to ground coordinates
$\mathbf{F}_i$	Forces & moment at the i-th hip
$\mathbf{f}_i$	Equation relating joint angles to the i-th hip position
$\mathbf{J}_i$	Jacobian relating joint velocities to the i-th hip velocity
$\mathbf{h}_i$	Equation relating joint angles to position of the i-th limb's center of mass
$\mathbf{J}_{lcg_i}$	Jacobian relating joint velocities to the velocity of the i-th limb's center of mass

$\mathbf{M}_{cg}$	Body's inertia tensor
m_{cg}	Body's mass
m_i	i-th limb's mass
q_i	i-th limb's joint values (only one joint, the foot in this formulation ...)
L	Leg length
$\mathbf{A}_i$	Limb's inertia tensor around the contact point
$\mathbf{r}_i$	Location of i-th hip with respect to the body's center of mass, in body coordinates
$\mathbf{r}_{cg \to h_i}$	Vector from the i-th limb's center of mass to the i-th hip
$\mathbf{r}_{cg \to f_i}$	Vector from the i-th limb's center of mass to its foot
$\mathbf{I}$	Identity matrix (3x3)
ϕ	Body angle with respect to vertical

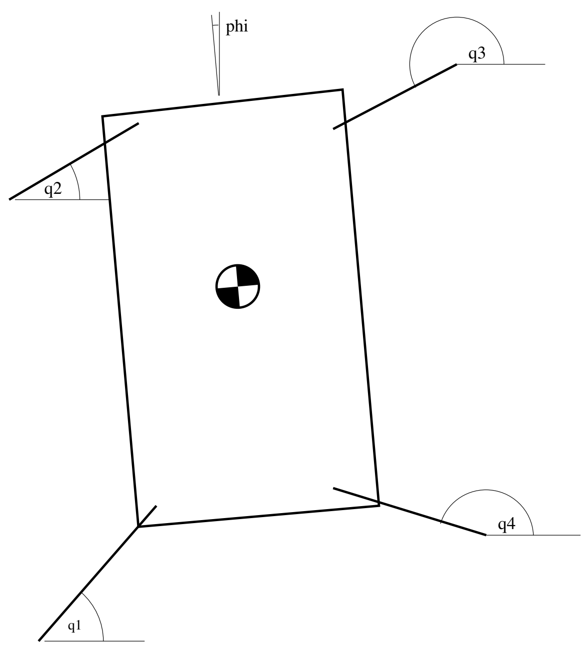

Figure 1: StickyBot Dorsal Model

$$
\begin{aligned}
{}^{b}\mathbf{F}_i &= \mathbf{K}_i\left({}^{b}\mathbf{x}_{hl_i} - {}^{b}\mathbf{x}_{hb_i}\right) && (1)\\
{}^{g}\mathbf{F}_i &= {}^{g}\mathbf{R}^{b}\,{}^{b}\mathbf{F}_i && (2)\\
{}^{g}\mathbf{R}^{b} &= \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} && (3)\\
\mathbf{x}_{hl_i} &= \mathbf{f}_i\left(q_i\right) && (4)\\
\mathbf{J}_i &= \frac{\partial \mathbf{f}_i}{\partial q_i} && (5)\\
\dot{\mathbf{x}}_{hl_i} &= \mathbf{J}_i\dot{\mathbf{q}}_i && (6)\\
\ddot{\mathbf{x}}_{hl_i} &= \dot{\mathbf{J}}_i\dot{\mathbf{q}}_i + \mathbf{J}_i\ddot{\mathbf{q}}_i && (7)\\
\mathbf{x}_{hb_i} &= \mathbf{x}_{cg} + \mathbf{r}_i && (8)\\
\ddot{\mathbf{x}}_{hb_i} &= \ddot{\mathbf{x}}_{cg} && (9)
\end{aligned}
$$

1.1 Equations of Motion

Newtonian moment balance around the contact point:

$$
\begin{aligned}
\mathbf{A}_i &= \left(\frac{1}{4}m_i L^2 + I_i\right) \qquad \text{(moment of inertia around foot)} && (10)\\
\mathbf{A}_i\ddot{q}_{i1} &= \mathbf{r}_{cgl_i} \times m_i\mathbf{g} - \mathbf{r}_{hl_i} \times \mathbf{F}_i - \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}\mathbf{F}_i && (11)
\end{aligned}
$$

Equation (11) is specific for this single joint limb formulation. More work required here to analyze a system with limbs that are multi-jointed.

Matrix and vector cross product equivalents:

$$
\begin{aligned}
\mathbf{r}\times &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -r_y & r_x & 0 \end{bmatrix} = \tilde{\mathbf{r}} \qquad \text{(planar cross product)} && (12)\\
\mathbf{r}\times &= \begin{bmatrix} -r_y & r_x & 1 \end{bmatrix} = \check{\mathbf{r}} \qquad \text{(scalar cross product with moment addition)} && (13)
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow \mathbf{A}_i\ddot{q}_{i1} &= m_i\check{\mathbf{r}}_{cgl_i}\mathbf{g} - \check{\mathbf{r}}_{hl_i}{}^{g}\mathbf{F}_i = 0 && (14)\\
\mathbf{A}_i\ddot{q}_{i1} &= m_i\check{\mathbf{r}}_{cgl_i}\mathbf{g} - \check{\mathbf{r}}_{hl_i}{}^{g}\mathbf{R}^{b}\,\mathbf{K}_i\left({}^{b}\mathbf{x}_{hl_i} - {}^{b}\mathbf{x}_{hb_i}\right) = 0 && (15)\\
\mathbf{A}_i\ddot{q}_{i1} &= m_i\check{\mathbf{r}}_{cgl_i}\mathbf{g} - \check{\mathbf{r}}_{hl_i}{}^{g}\mathbf{R}^{b}\,\mathbf{K}_i\left({}^{g}\mathbf{R}^{b^{-1}}\mathbf{x}_{hl_i} - {}^{g}\mathbf{R}^{b^{-1}}\mathbf{x}_{cg} - \mathbf{r}_i\right) && (16)
\end{aligned}
$$

From Newtonian force balance on body:

$$\mathbf{M}_{cg}\ddot{\mathbf{x}}_{cg} = m_{cg}\mathbf{g} + \Sigma^{g}\mathbf{F}_i + \Sigma\left({}^{b}\mathbf{r}_i \times {}^{b}\mathbf{F}_i\right) \tag{17}$$

(Taking advantage of the fact that the z-axes are the same in the body and ground frames)

$$\mathbf{M}_{cg}\ddot{\mathbf{x}}_{cg} = m_{cg}\mathbf{g} + \Sigma\left(\left[{}^{g}\mathbf{R}^{b} + {}^{b}\tilde{\mathbf{r}}_i\right]\left[\mathbf{K}_i\,{}^{b}\mathbf{R}^{g}\left(\mathbf{x}_{hl_i} - \mathbf{x}_{cg}\right) - \mathbf{K}_i\mathbf{r}_i\right]\right) \tag{18}$$

We now have acceleration equations for each unknown.

Next step is to compute the extensions and wall reaction forces. Figure out a strategy to accomplish desired motion, and minimize/equalize wall reaction forces.

Connection Extensions:

$$\begin{aligned} {}^{b}\mathbf{x}_{hl_i} - {}^{b}\mathbf{x}_{hb_i} &= {}^{g}\mathbf{R}^{b^{-1}}\left({}^{g}\mathbf{x}_{hl_i} - {}^{g}\mathbf{x}_{cg}\right) - {}^{b}\mathbf{r}_i && (19)\\ &= {}^{g}\mathbf{R}^{b^{-1}}\left(\mathbf{f}_i(q_i) - {}^{g}\mathbf{x}_{cg}\right) - {}^{b}\mathbf{r}_i && (20) \end{aligned}$$

Reaction forces are found by taking the sum of forces and moments around each limb's center of mass and using the accelerations of the center of mass, previously determined:

$$\begin{aligned} \mathbf{M}_{lcg_i}\ddot{\mathbf{x}}_{lcg_i} &= -\mathbf{F}_i + \mathbf{F}_{r_i} + m_i\mathbf{g} - \tilde{\mathbf{r}}_{cg\to h_i}\mathbf{F}_i + \tilde{\mathbf{r}}_{cg\to f_i}\mathbf{F}_{r_i} && (21)\\ &= -\left(\mathbf{I} + \tilde{\mathbf{r}}_{cg\to h_i}\right)\mathbf{F}_i + m_i\mathbf{g} + \left(\mathbf{I} + \tilde{\mathbf{r}}_{cg\to f_i}\right)\mathbf{F}_{r_i} && (22)\\ \Rightarrow \left(\mathbf{I} + \tilde{\mathbf{r}}_{cg\to f_i}\right)\mathbf{F}_{r_i} &= \mathbf{M}_{lcg_i}\ddot{\mathbf{x}}_{lcg_i} + \left(\mathbf{I} + \tilde{\mathbf{r}}_{cg\to h_i}\right)\mathbf{F}_i - m_i\mathbf{g} && (23) \end{aligned}$$

Since there are only reaction forces (no moments) at the foot contact, the cross product portions drop out, which leaves:

$$\mathbf{F}_{r_i} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\left(\mathbf{M}_{lcg_i}\ddot{\mathbf{x}}_{lcg_i} + \mathbf{F}_i - m_i\mathbf{g}\right) \tag{24}$$

$$\mathbf{F}_{r_i} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\left(\mathbf{M}_{lcg_i}\left(\dot{\mathbf{J}}_{li}\dot{\mathbf{q}}_i + \mathbf{J}_{li}\ddot{\mathbf{q}}_i\right) + \mathbf{F}_i - m_i\mathbf{g}\right) \tag{25}$$

where $\mathbf{J}_{li}$ is the jacobian that relates the joint velocity to the cartesian velocity of the limb's center of mass.

Expanding out $\mathbf{M}_{lcg_i}\ddot{\mathbf{x}}_{lcg_i}$ yields:

$$\begin{aligned} \mathbf{M}_{lcg_i}\ddot{\mathbf{x}}_{lcg_i} &= \mathbf{D}_i + \mathbf{E}_i\mathbf{r}_i && (26)\\ \mathbf{D}_i &= \mathbf{M}_{lcg_i}\dot{\mathbf{J}}_{li}\dot{\mathbf{q}}_i + \mathbf{M}_{lcg_i}\mathbf{J}_{li}\mathbf{A}_i^{-1}\left[m_i\check{\mathbf{r}}_{cgl_i}\mathbf{g} - \check{\mathbf{r}}_{hl_i}{}^{g}\mathbf{R}^{b}\,\mathbf{K}_i\,{}^{b}\mathbf{R}^{g}\left(\mathbf{x}_{hl_i} - \mathbf{x}_{cg}\right)\right] && (27)\\ \mathbf{E}_i &= \mathbf{M}_{lcg_i}\mathbf{J}_{li}\mathbf{A}_i^{-1}\check{\mathbf{r}}_{hl_i}{}^{g}\mathbf{R}^{b}\,\mathbf{K}_i && (28) \end{aligned}$$

Expanding out $\ddot{\mathbf{q}}_i$ and $\mathbf{F}_i$ yields:

$$\mathbf{F}_{r_i} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \left\{ \mathbf{D}_i - m_i \mathbf{g} + {}^g\mathbf{R}^b \, \mathbf{K}_i \, {}^b\mathbf{R}^g \left(\mathbf{x}_{hl_i} - \mathbf{x}_{cg} \right) + \left(\mathbf{E}_i - {}^g\mathbf{R}^b \, \mathbf{K}_i \right) \mathbf{r}_i \right\} \tag{29}$$

Remember that the current method of control is modifying $\mathbf{r}_{i\phi}$, so let's get everything in terms of r_i.

$$\mathbf{F}_{r_i} = \mathbf{A}_i + \mathbf{C}_i \mathbf{r}_i \tag{30}$$

where:

$$\mathbf{A}_i = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \left\{ \mathbf{D}_i - m_i \mathbf{g} + {}^g\mathbf{R}^b \, \mathbf{K}_i \, {}^b\mathbf{R}^g \left(\mathbf{x}_{hl_i} - \mathbf{x}_{cg} \right) \right\} \tag{31}$$

$$\mathbf{C}_i = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \left\{ \mathbf{E}_i - {}^g\mathbf{R}^b \, \mathbf{K}_i \right\} \tag{32}$$

Pulling $\mathbf{r}_i$ out of the equation (18) is possible if you're really only interested in changing $r_{i\phi}$, which does not show up in $\tilde{\mathbf{r}}_i$:

$$\ddot{\mathbf{x}}_{cg} = \mathbf{G} + \Sigma \mathbf{H}_i \mathbf{r}_i \tag{33}$$

where:

$$\mathbf{G} = \mathbf{M}_{cg}^{-1} \left[m_{cg} \mathbf{g} + \Sigma \left(\left[{}^g\mathbf{R}^b + {}^b\tilde{\mathbf{r}}_i \right] \mathbf{K}_i \, {}^b\mathbf{R}^g \left(\mathbf{x}_{hl_i} - \mathbf{x}_{cg} \right) \right) \right] \tag{34}$$

$$\mathbf{H}_i = \mathbf{M}_{cg}^{-1} \left[{}^g\mathbf{R}^b + {}^b\tilde{\mathbf{r}}_i \right] \mathbf{K}_i \tag{35}$$

So now, you have N controls that yield equations (30) and (33) which may allow us to answer these questions:

- How can one equalize the $|\mathbf{F}_r|$'s while maintaining $\ddot{\mathbf{x}}_{cg}$ (or $\dot{\mathbf{x}}_{cg}$)?
- What is the limit of $\ddot{\mathbf{x}}_{cg}$ (or $\dot{\mathbf{x}}_{cg}$) for a given limit on $|\mathbf{F}_r|$?

Go back and look at increasing the number of joints per limb and modifying/generalizing the hip compliance assumption?

To start, I'll need to do a visualization routine for the StickyBot model, and run a simulation for a generic gait. Then, look at implementing a reaction force reduction controller on that simple gait.

1.1.1 Adding Damping

In order for a gait simulation to have a chance at being stable, joint damping must be included. This is accomplished by estimating a damping constant that will make the sytem approximately damped. This is done by assuming a damping ratio of one for each axis in each joint, using $1/N$ of the body's mass and rotational inertia for the prismatic and rotational joints, respectively.

$$b_{i_{x,y}} = 2\sqrt{\frac{m_{cg}\, k_{i_{x,y}}}{N}} \tag{36}$$

$$b_{i_\phi} = 2\sqrt{\frac{I_{cg}\, k_{i_\phi}}{N}} \tag{37}$$

$${}^{b}\mathbf{F}_i = \mathbf{K}_i \left({}^{b}\mathbf{x}_{hl_i} - {}^{b}\mathbf{x}_{hb_i}\right) + \mathbf{B}_i \left({}^{b}\dot{\mathbf{x}}_{hl_i} - {}^{b}\dot{\mathbf{x}}_{hb_i}\right) \tag{38}$$

$$= \mathbf{K}_i \left({}^{b}\mathbf{R}^{g} \left({}^{g}\mathbf{x}_{hl_i} - {}^{g}\mathbf{x}_{cg}\right) - {}^{b}\mathbf{r}_i\right) + \mathbf{B}_i \left({}^{b}\mathbf{R}^{g} \left({}^{g}\dot{\mathbf{x}}_{hl_i} - {}^{g}\dot{\mathbf{x}}_{cg}\right) + {}^{b}\dot{\mathbf{R}}^{g} \left({}^{g}\mathbf{x}_{hl_i} - {}^{g}\mathbf{x}_{cg}\right) - \dot{\mathbf{r}}_i\right) \tag{39}$$

$${}^{g}\mathbf{F}_i = {}^{g}\mathbf{R}^{b}\, {}^{b}\mathbf{F}_i \tag{40}$$

$${}^{g}\mathbf{F}_i = \mathbf{L}_i + \mathbf{N}_i\, \mathbf{r}_i + \mathbf{P}_i\, \dot{\mathbf{r}}_i \tag{41}$$

$$\mathbf{L}_i = {}^{g}\mathbf{R}^{b} \left(\mathbf{K}_i\, {}^{b}\mathbf{R}^{g} \left({}^{g}\mathbf{x}_{hl_i} - {}^{g}\mathbf{x}_{cg}\right) + \mathbf{B}_i \left({}^{b}\mathbf{R}^{g} \left({}^{g}\dot{\mathbf{x}}_{hl_i} - {}^{g}\dot{\mathbf{x}}_{cg}\right) + {}^{b}\dot{\mathbf{R}}^{g} \left({}^{g}\mathbf{x}_{hl_i} - {}^{g}\mathbf{x}_{cg}\right)\right)\right) \tag{42}$$

$$\mathbf{N}_i = -{}^{g}\mathbf{R}^{b}\, \mathbf{K}_i \tag{43}$$

$$\mathbf{P}_i = -{}^{g}\mathbf{R}^{b}\, \mathbf{B}_i \tag{44}$$

Adding to equation (30), the reaction force is now:

$$\mathbf{F}_{r_i} = \mathbf{A}_i + \mathbf{C}_i\, \mathbf{r}_i + \mathbf{S}_i\, \dot{\mathbf{r}}_i \tag{45}$$

$$\mathbf{A}_i = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \left(\mathbf{D}_i + \mathbf{L}_i - m_i \mathbf{g}\right) \tag{46}$$

$$\mathbf{S}_i = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \mathbf{P}_i \tag{47}$$

$\mathbf{C}_i$ is the same as before.

1.2 Effect of r_{i_ϕ} on Reaction Forces and Body Acceleration

The effect of the "control" on the reaction forces is found by taking the partial of the reaction forces with respect to r_{i_ϕ}. This yields:

$$\frac{\partial \mathbf{F}_{r_i}}{\partial r_{i_\phi}} = \mathbf{C}_i \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{48}$$

$$\left|\frac{\partial \mathbf{F}_{r_i}}{\partial r_{i_\phi}}\right| = \left(\begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \mathbf{C}_i^{\mathrm{T}}\, \mathbf{C}_i \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right)^{\frac{1}{2}} = \frac{L}{2}\, \frac{m_i}{A_i}\, k_\phi \tag{49}$$

$$\frac{\partial \mathbf{F}_{r_i}}{\partial \dot{r}_{i_\phi}} = \mathbf{S}_i \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{50}$$

$$\left| \frac{\partial \mathbf{F}_{r_i}}{\partial \dot{r}_{i_\phi}} \right| = \left(\begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \mathbf{S}_i^{\mathrm{T}} \mathbf{S}_i \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right)^{\frac{1}{2}} = \frac{L}{2} \frac{m_i}{A_i} b_\phi \tag{51}$$

This is ignoring the effect of ankle angle on the reaction force, which is dependent on r_{i_ϕ} ...

As for body acceleration, Equation (33) is modified with the addition of damping at the hip:

$$\ddot{\mathbf{x}}_{cg} = \mathbf{G} + \Sigma \mathbf{H}_i \mathbf{r}_i + \Sigma \mathbf{T}_i \dot{\mathbf{r}}_i \tag{52}$$

where:

$$\mathbf{G} = \mathbf{M}_{cg}^{-1} m_{cg} \mathbf{g} + \mathbf{M}_{cg}^{-1} \Sigma \left(\mathbf{I} + {}^b\tilde{\mathbf{r}}_i {}^b\mathbf{R}^g \right) \mathbf{L}_i \qquad (\mathbf{L}_i \text{ defined in equation (42)}) \tag{53}$$

$$\mathbf{H}_i = \mathbf{M}_{cg}^{-1} \left(\mathbf{I} + {}^b\tilde{\mathbf{r}}_i {}^b\mathbf{R}^g \right) \mathbf{N}_i \qquad (\mathbf{N}_i \text{ defined in equation (43)}) \tag{54}$$

$$\mathbf{T}_i = \mathbf{M}_{cg}^{-1} \left(\mathbf{I} + {}^b\tilde{\mathbf{r}}_i {}^b\mathbf{R}^g \right) \mathbf{P}_i \qquad (\mathbf{P}_i \text{ defined in equation (44)}) \tag{55}$$

Changes of the body's acceleration with respect to changes in control are:

$$\frac{\partial \ddot{\mathbf{x}}_{cg}}{\partial r_{i_\phi}} = \mathbf{H}_i \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = -\frac{k_\phi}{I_{cg}} \tag{56}$$

$$\frac{\partial \ddot{\mathbf{x}}_{cg}}{\partial \dot{r}_{i_\phi}} = \mathbf{T}_i \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = -\frac{b_\phi}{I_{cg}} \tag{57}$$

This is ignoring the effect of ankle angle on the body's acceleration, which is dependent on r_{i_ϕ} ...

1.3 Some Specifics

Starting with the limb equations:

$$\mathbf{x}_{hl_i} = \begin{bmatrix} L\cos q_{i1} \\ L\sin q_{i1} \\ q_{i1} \end{bmatrix} \tag{58}$$

$$\dot{\mathbf{x}}_{hl_i} = \begin{bmatrix} -L\sin q_{i1} \\ L\cos q_{i1} \\ 1 \end{bmatrix} \dot{q}_{i1} \tag{59}$$

$$\Rightarrow \mathbf{J}_i = \begin{bmatrix} -L\sin q_{i1} \\ L\cos q_{i1} \\ 1 \end{bmatrix} \tag{60}$$

$$\dot{\mathbf{J}}_i = \begin{bmatrix} -L\cos q_{i1} \\ -L\sin q_{i1} \\ 0 \end{bmatrix} \dot{q}_{i1} \tag{61}$$

$$\ddot{\mathbf{x}}_{hl_i} = \begin{bmatrix} -L\cos q_{i1} \\ -L\sin q_{i1} \\ 0 \end{bmatrix} \dot{q}_{i1}^2 + \begin{bmatrix} -L\sin q_{i1} \\ L\cos q_{i1} \\ 1 \end{bmatrix} \ddot{q}_{i1} \tag{62}$$