3.1 TRANSFORMATION OF COORDINATES

Any vector $\mathbf{r}$ can be resolved in one or more systems of coordinates. In many problems of dynamics relations between the components of $\mathbf{r}$ in various coordinate systems prove extremely useful. To derive such relations, let us consider Fig. 3.1 and write the vector $\mathbf{r}$ in terms of components along two rectangular sets of axes, x_i $(i = 1, 2, 3)$ and ξ_i $(i = 1, 2, 3)$. The unit vectors along these axes are denoted by $\mathbf{i}, \mathbf{j}, \mathbf{k}$ and $\mathbf{i}', \mathbf{j}', \mathbf{k}'$, respectively, so that

$$\mathbf{r} = x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k} = \xi_1\mathbf{i}' + \xi_2\mathbf{j}' + \xi_3\mathbf{k}'. \tag{3.1}$$

Since these are two ways of expressing the same vector, the components x_i and ξ_i must evidently be related. The relation between ξ_1 and the components x_i can be obtained by writing the scalar product of $\mathbf{r}$ and $\mathbf{i}'$ with the result

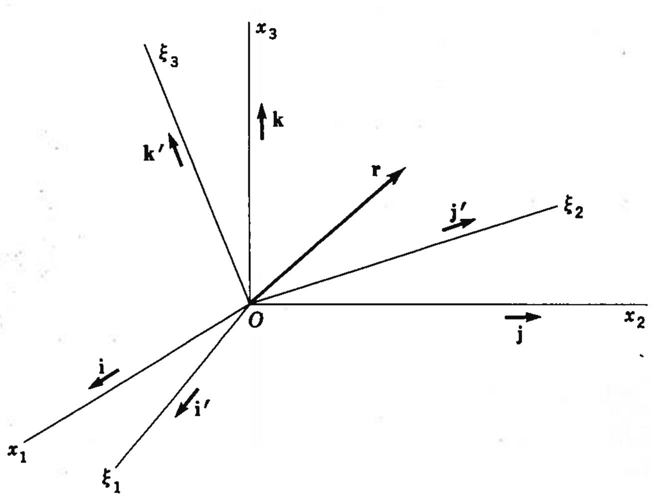
$$\begin{aligned}\xi_1 &= (\mathbf{i}'\cdot\mathbf{i})x_1 + (\mathbf{i}'\cdot\mathbf{j})x_2 + (\mathbf{i}'\cdot\mathbf{k})x_3\\ &= x_1\cos(\xi_1,x_1) + x_2\cos(\xi_1,x_2) + x_3\cos(\xi_1,x_3)\\ &= l_{11}x_1 + l_{12}x_2 + l_{13}x_3,\end{aligned} \tag{3.2}$$

where $l_{1j} = \cos(\xi_1,x_j)$ $(j = 1, 2, 3)$ are the direction cosines between axis ξ_1 and axes x_j. Similarly, we can express ξ_2 and ξ_3 in terms of the x_j components so that Eq. (3.2) can be generalized to

$$\xi_i = l_{i1}x_1 + l_{i2}x_2 + l_{i3}x_3 = \sum_{j=1}^{3} l_{ij}x_j, \qquad i = 1, 2, 3. \tag{3.3}$$

Next let us define the 3×1 column matrices $\{x\} = \{x_j\}$ and $\{\xi\} = \{\xi_i\}$ representing the vector $\mathbf{r}$ in terms of the corresponding components, as well

FIGURE 3.1



as the 3×3 square matrix $[l] = [l_{ij}]$ of the direction cosines, and write Eq. (3.3) in the matrix form

$$\{\xi\} = [l]\{x\}, \tag{3.4}$$

where the matrix $[l]$ may be regarded as an operator transforming the vector $\{x\}$ into the vector $\{\xi\}$. Equation (3.4) represents a *coordinate transformation* between two cartesian sets of axes. As such, it is not the most general type of coordinate transformation; a more general one would be the transformation between cartesian coordinates and generalized coordinates given by Eq. (2.81). In the case of the *linear transformation*

$$\{y\} = [a]\{x\}, \tag{3.5}$$

in which $[a]$ is a square matrix of coefficients a_{ij}, we can obtain the vector $\{x\}$ by premultiplying both sides of Eq. (3.5) by the reciprocal $[a]^{-1}$ of $[a]$

$$\{x\} = [a]^{-1}\{y\}, \tag{3.6}$$

provided that the matrix $[a]$ is not singular. In the special case of the transformation matrix $[l]$, however, the coefficients l_{ij} are not all independent. To show this, we can write the scalar products of $\mathbf{r}$ and $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ in sequence and obtain the relation

$$x_r = \sum_{s=1}^{3} l_{sr}\xi_s, \qquad r = 1, 2, 3, \tag{3.7}$$

which assumes the matrix form

$$\{x\} = [l]^T\{\xi\}, \tag{3.8}$$

where $[l]^T$, defined by $[l_{ij}]^T = [l_{ji}]$ $(i, j = 1, 2, 3)$, denotes the transpose of the matrix $[l]$. Introducing Eq. (3.4) into (3.8), we arrive at

$$[l]^T[l] = [1] = [\delta_{ij}], \tag{3.9}$$

where $[1]$ is the *identity matrix* or *unit matrix*, namely, a matrix with all its elements equal to the *Kronecker delta* defined by

$$\delta_{ij} = \begin{cases}1 & i = j,\\ 0 & i \neq j.\end{cases} \tag{3.10}$$

From Eq. (3.9), we can easily conclude that the matrix of the direction cosines l_{ij} satisfies the relation

$$[l]^{-1} = [l]^T, \tag{3.11}$$

which implies that the reciprocal, or inverse, of $[l]$ is equal to the transpose of $[l]$. A transformation satisfying relation (3.11) is called an *orthonormal* *transformation.* Equation (3.9) can be written in index notation as

$$\sum_{k=1}^{3} l_{ki}l_{kj} = \delta_{ij}, \qquad i, j = 1, 2, 3, \tag{3.12}$$

which expresses the fact that axes x_i on the one hand and axes ξ_j on the other form *orthogonal sets of axes* and also that the length of the vector $\mathbf{r}$ is the same regardless of the set of axes in which it is resolved. The latter statement can be written

$$\mathbf{r}\cdot\mathbf{r} = \{x\}^T\{x\} = \{\xi\}^T\{\xi\}, \tag{3.13}$$

which can also be used to derive Eq. (3.9).

It may prove of interest to calculate the value of the determinant $|l|$ of the matrix $[l]$. From matrix algebra we have the relation

$$|[a][b]| = |a|\,|b|, \tag{3.14}$$

or the determinant of a product of two matrices is equal to the product of the determinants of the two matrices. But the determinant of the identity matrix is equal to 1, and since the determinant of a matrix is equal to the determinant of the transposed matrix, it follows from Eqs. (3.9) and (3.14) that

$$|l|^2 = 1. \tag{3.15}$$

Equation (3.15) indicates that $|l|$ may assume the value $+1$ or -1. For the type of transformations we shall be concerned with the value is $+1$ (see Sec. 4.1).

The matrix $[l]$ can be regarded as being the result of three successive rotations leading from system $\{x\}$ to system $\{\xi\}$, as we are going to see in the next section.

3.2 ROTATING COORDINATE SYSTEMS

In many dynamical problems involving spinning bodies it is convenient to express the motion in terms of components along rotating frames of reference, which, by definition, are noninertial frames. If this motion is to be related to the inertial space again, we must produce expressions relating the components of the rotating and the fixed systems of axes. In doing so, we recognize that one of the coordinate systems of Sec. 3.1 may be regarded as inertial and the other one as rotating in space, in which case the associated direction cosines are implicit functions of time. In the sequel we shall develop explicit expressions for the direction cosines between an inertial set of axes, say x_i, and a rotating one, denoted by ξ_i. The latter is obtained from the former by means of three successive rotations θ_1, θ_2, and θ_3 about axes x_1, y_2, and z_3, resulting in the systems y_i, z_i, and ξ_i, respectively.

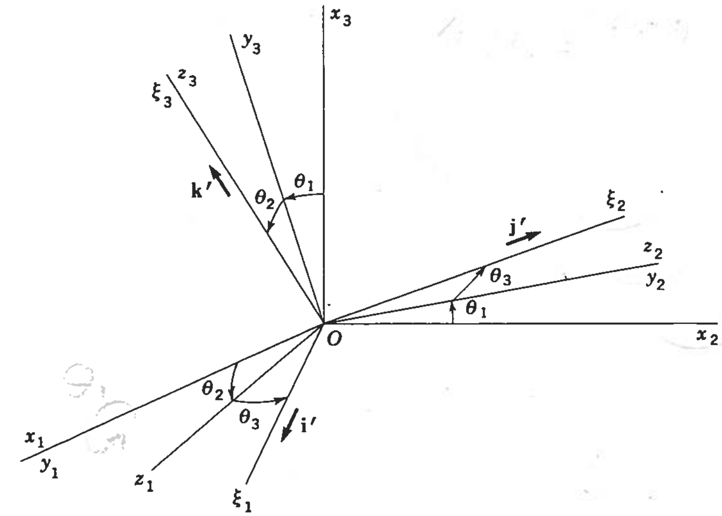


FIGURE 3.2

From Fig. 3.2 we conclude that the relation between the systems of coordinates x_i and y_i is as follows:

$$\begin{aligned} y_1 &= x_1, \\ y_2 &= x_2\cos\theta_1 + x_3\sin\theta_1, \\ y_3 &= -x_2\sin\theta_1 + x_3\cos\theta_1, \end{aligned} \tag{3.16}$$

which can be written in matrix form as

$$\begin{Bmatrix} y_1 \\ y_2 \\ y_3 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_1 & \sin\theta_1 \\ 0 & -\sin\theta_1 & \cos\theta_1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} \tag{3.17}$$

or in more compact notation as

$$\{y\} = [R_1(\theta_1)]\{x\}. \tag{3.18}$$

The *rotation matrix* $[R_1(\theta_1)]$, denoting the square matrix of the coefficients in Eq. (3.17), represents the rotation of a system of axes originally coincident with axes x_i by an angle θ_1 about axis x_1. In a similar fashion, we can write

$$\{z\} = \begin{bmatrix} \cos\theta_2 & 0 & -\sin\theta_2 \\ 0 & 1 & 0 \\ \sin\theta_2 & 0 & \cos\theta_2 \end{bmatrix} \{y\} = [R_2(\theta_2)]\{y\} \tag{3.19}$$

and

$$\{\xi\} = \begin{bmatrix} \cos\theta_3 & \sin\theta_3 & 0 \\ -\sin\theta_3 & \cos\theta_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \{z\} = [R_3(\theta_3)]\{z\}. \tag{3.20}$$

Combining Eqs. (3.18) to (3.20), we obtain

$$\{\xi\} = [R_3(\theta_3)][R_2(\theta_2)][R_1(\theta_1)]\{x\} = [l]\{x\}, \tag{3.21}$$

where the matrix of direction cosines has the form

$$[l] = \begin{bmatrix} c\theta_2\, c\theta_3 & c\theta_1\, s\theta_3 + s\theta_1\, s\theta_2\, c\theta_3 & s\theta_1\, s\theta_3 - c\theta_1\, s\theta_2\, c\theta_3 \\ -c\theta_2\, s\theta_3 & c\theta_1\, c\theta_3 - s\theta_1\, s\theta_2\, s\theta_3 & s\theta_1\, c\theta_3 + c\theta_1\, s\theta_2\, s\theta_3 \\ s\theta_2 & -s\theta_1\, c\theta_2 & c\theta_1\, c\theta_2 \end{bmatrix}, \tag{3.22}$$

in which the abbreviations $s\theta_i = \sin\theta_i$ and $c\theta_i = \cos\theta_i$ are used to save space.

Next we wish to explore the possibility of expressing *finite rotations* as vector quantities. The natural thing to investigate is whether the preceding rotations can be represented as vectors $\boldsymbol{\theta}_i$ $(i = 1, 2, 3)$ directed along the axes x_1, y_2, and z_3, respectively. For this purpose, let us check whether the rotations satisfy the addition rule for vectors, namely, that addition is a commutative process. This is equivalent to requiring that the order of the rotations be immaterial. The check can be made by using the rotation matrices developed above and going from system $\{x\}$ to system $\{z\}$ in two ways: (1) according to the sequence given by Eqs. (3.18) and (3.19) and (2) by a rotation θ_2 about axis x_2 followed by a rotation θ_1 about axis y_1. If the results obtained in the two ways are equal, it can be concluded that finite rotations are commutative and satisfy at least the addition rule for vectors. The two different sequences yield

$$\{z\} = [R_2(\theta_2)][R_1(\theta_1)]\{x\} = \begin{bmatrix} c\theta_2 & s\theta_1\, s\theta_2 & -c\theta_1\, s\theta_2 \\ 0 & c\theta_1 & s\theta_1 \\ s\theta_2 & -s\theta_1\, c\theta_2 & c\theta_1\, c\theta_2 \end{bmatrix} \{x\} \tag{3.23}$$

and

$$\{z\} = [R_1(\theta_1)][R_2(\theta_2)]\{x\} = \begin{bmatrix} c\theta_2 & 0 & -s\theta_2 \\ s\theta_1\, s\theta_2 & c\theta_1 & s\theta_1\, c\theta_2 \\ c\theta_1\, s\theta_2 & -s\theta_1 & c\theta_1\, c\theta_2 \end{bmatrix} \{x\}, \tag{3.24}$$

and it is not difficult to see that the addition rule for vectors is violated, as the two resulting vectors $\{z\}$ are not the same. This is not at all surprising since, in general, matrix products are not commutative

$$[R_2(\theta_2)][R_1(\theta_1)] \neq [R_1(\theta_1)][R_2(\theta_2)]. \tag{3.25}$$

Hence, *finite angles of rotation cannot be represented by vectors.*

Fortunately, however, considerable interest remains in representing *infinitesimal rotations*, rather than finite rotations, by vectors. Indeed, in the particular case in which the angles of rotation are sufficiently small to permit higher-order terms to be ignored Eqs. (3.23) and (3.24) both yield the same result. Hence, following this line of thought, we can show that infinitesimal rotations can be represented by vectors. As we are not so much interested in representing rotations by vectors as in representing rates of change of rotations, namely, angular velocities, by vectors, we shall find it to our advantage to change the approach slightly.

Let us consider a vector $\mathbf{r}$ fixed with respect to a set of moving axes ξ_i. Because the set of axes ξ_i rotates relative to an inertial space, say the set x_i, the vector $\mathbf{r}$ undergoes some change. We may recall that a change in the direction of a vector is sufficient to bring about a change in the vector, and our interest lies in calculating the rate of change of $\mathbf{r}$ due to the angular velocity of the reference frame ξ_i. We have the choice of expressing $\mathbf{r}$ and the rate of change of $\mathbf{r}$ in terms of components along the inertial system x_i or along the moving system ξ_i. This may sound like a paradox in view of the fact that the vector $\mathbf{r}$ is fixed relative to the system ξ_i. The fact remains, however, that, due to the rotation of the system ξ_i, the vector $\mathbf{r}$ changes continuously with respect to an inertial space and the vector representing the corresponding rate of change can be resolved into components along the set x_i or the set ξ_i. It turns out that in the study of spinning bodies it is frequently more useful to express the motion in terms of components along the moving system ξ_i. To accomplish this, let us assume that the angles of rotation $\Delta\theta_i$ are sufficiently small for the approximations $\sin\Delta\theta_i \approx \Delta\theta_i$ and $\cos\Delta\theta_i \approx 1$ $(i = 1, 2, 3)$ to be justified. Then premultiplying both sides of Eq. (3.21) by $[l]^T$, we arrive at

$$\{x\} = \{\xi\} + \{\Delta\xi\} = [l]^T\{\xi\} = \begin{bmatrix} 1 & -\Delta\theta_3 & \Delta\theta_2 \\ \Delta\theta_3 & 1 & -\Delta\theta_1 \\ -\Delta\theta_2 & \Delta\theta_1 & 1 \end{bmatrix} \{\xi\}$$

$$= [1]\{\xi\} + \begin{bmatrix} 0 & -\Delta\theta_3 & \Delta\theta_2 \\ \Delta\theta_3 & 0 & -\Delta\theta_1 \\ -\Delta\theta_2 & \Delta\theta_1 & 0 \end{bmatrix} \{\xi\}, \tag{3.26}$$

where $[l]^T$ was obtained by transposing Eq. (3.22) and letting $\theta_i \to \Delta\theta_i$ $(i = 1, 2, 3)$. Equation (3.26) leads to

$$\{\Delta\xi\} = [\Delta\theta]\{\xi\}, \tag{3.27}$$

where

$$[\Delta\theta] = \begin{bmatrix} 0 & -\Delta\theta_3 & \Delta\theta_2 \\ \Delta\theta_3 & 0 & -\Delta\theta_1 \\ -\Delta\theta_2 & \Delta\theta_1 & 0 \end{bmatrix} \tag{3.28}$$

is a *skew-symmetric*, or *antisymmetric*, matrix. Hence, a vector $\mathbf{r}$, fixed with respect to a moving system of coordinates ξ_i, will undergo an incremental change $\Delta\mathbf{r}$ relative to an inertial space as a result of an incremental rotation with components $\Delta\theta_i$ $(i = 1, 2, 3)$ of the moving system about axes ξ_i, respectively. In terms of components along the inertial system, the vector $\mathbf{r} + \Delta\mathbf{r}$ is given by $\{x\}$, whereas in terms of components along the system ξ_i, it is expressed by $\{\xi\} + \{\Delta\xi\}$, where the column matrix $\{\Delta\xi\}$ represents the increment $\Delta\mathbf{r}$. All quantities in Eqs. (3.27) and (3.28) are in terms of components along the moving system ξ_i. Clearly, the results (3.27) and (3.28) do not depend on the order of the rotations if the rotations are small. Dividing Eqs. (3.27) and (3.28) by the associated time increment Δt and letting $\Delta t \to 0$, we obtain the time derivative

$$\{\dot{\xi}\} = \lim_{\Delta t\to 0}\left\{\frac{\Delta\xi}{\Delta t}\right\} = \lim_{\Delta t\to 0}\left[\frac{\Delta\theta}{\Delta t}\right]\{\xi\} = [\omega]\{\xi\}, \tag{3.29}$$

where $[\omega]$ is the skew-symmetric matrix

$$[\omega] = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}, \tag{3.30}$$

in which ω_i $(i = 1, 2, 3)$ are the angular velocity components of the moving system ξ_i relative to the inertial space x_i when expressed in terms of components along the system ξ_i.

It is easy to show that the vector counterpart of Eq. (3.29) is

$$\dot{\mathbf{r}} = \boldsymbol{\omega} \times \mathbf{r}, \tag{3.31}$$

where

$$\mathbf{r} = \xi_1\mathbf{i}' + \xi_2\mathbf{j}' + \xi_3\mathbf{k}' \tag{3.32}$$

and

$$\boldsymbol{\omega} = \omega_1\mathbf{i}' + \omega_2\mathbf{j}' + \omega_3\mathbf{k}'. \tag{3.33}$$

Equation (3.31) can easily be interpreted physically by means of Fig. 3.3, in which $\mathbf{r}$ represents the position vector, $\boldsymbol{\omega}$ the angular velocity vector which is coincident with the instantaneous axis of rotation, and $\dot{\mathbf{r}}$ a vector normal to both $\mathbf{r}$ and $\boldsymbol{\omega}$ and in the direction shown.

As a special application of Eq. (3.31), we can obtain expressions for the time derivatives of the unit vectors $\mathbf{i}'$, $\mathbf{j}'$, and $\mathbf{k}'$, respectively, in the form

$$\begin{aligned} \frac{d\mathbf{i}'}{dt} &= \boldsymbol{\omega}\times\mathbf{i}' = \omega_3\mathbf{j}' - \omega_2\mathbf{k}', \\ \frac{d\mathbf{j}'}{dt} &= \boldsymbol{\omega}\times\mathbf{j}' = \omega_1\mathbf{k}' - \omega_3\mathbf{i}', \\ \frac{d\mathbf{k}'}{dt} &= \boldsymbol{\omega}\times\mathbf{k}' = \omega_2\mathbf{i}' - \omega_1\mathbf{j}'. \end{aligned} \tag{3.34}$$

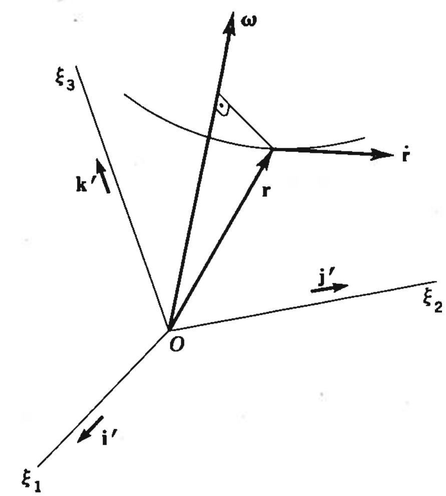


FIGURE 3.3

In fact, Eq. (3.31) can be interpreted as the time derivative of $\mathbf{r}$, in which the magnitudes ξ_1, ξ_2, and ξ_3 are constant in time and the unit vectors have time derivatives according to Eqs. (3.34).

In the above discussion, although not specifically stated, it was implied that the elements of the skew-symmetric matrix $[\omega]$ form a vector $\boldsymbol{\omega}$ in all cartesian coordinate systems. For $\boldsymbol{\omega}$ to qualify as a vector, however, it must also transform like the components of a vector, which has yet to be shown. In the sequel we shall examine this question.

The relation between the components of the vector $\boldsymbol{\omega}$ and the matrix $[\omega]$ can be written in the form

$$\omega_i = \frac{1}{2}\sum_{j=1}^{3}\sum_{k=1}^{3}\epsilon_{ijk}\omega_{kj}, \tag{3.35}$$

in which ϵ_{ijk} is the standard *epsilon symbol*, defined to be equal to zero if any two of the three indices are equal, equal to $+1$ if the indices are in cyclic order, and equal to -1 if they are not. The inverse relation corresponding to Eq. (3.35) is

$$\omega_{nm} = \sum_{l=1}^{3}\epsilon_{mnl}\omega_l. \tag{3.36}$$

For $\boldsymbol{\omega}$ to qualify as a vector, it must be shown that for an orthonormal transformation defined by the matrix $[a]$, $[a]^T = [a]^{-1}$, relating the elements

ω_{nm} to the elements ω'_{kj} of the transformed matrix by

$$\omega'_{kj} = \sum_{m=1}^{3} \sum_{n=1}^{3} a_{kn}\omega_{nm}a_{jm}, \tag{3.37}$$

the components in the new coordinate system have the form

$$\omega'_i = \sum_{j=1}^{3} a_{ij}\omega_j. \tag{3.38}$$

It turns out that the coordinate transformation in question has the form

$$\omega'_i = |a| \sum_{j=1}^{3} a_{ij}\omega_j, \tag{3.39}$$

where $|a|$ is the determinant of the matrix $[a]$, rather than the form (3.38) (see Ref. 1, p. 130). Thus for $\boldsymbol{\omega}$ to qualify as a vector, $|a|$ must be equal to $+1$. This actually happens only in the case of an orthonormal transformation corresponding to a *proper rotation* defined by a transformation matrix with the determinant equal to $+1$, as opposed to the *improper rotation* with the determinant equal to -1. An orthonormal transformation corresponding to a proper rotation transforms a right-handed system into another right-handed system, as in the case described by Eq. (3.4). Quantities transforming according to Eq. (3.39) are called *pseudovectors* or *axial vectors*. Hence, the elements of any 3×3 skew-symmetric matrix form the components of a pseudovector. For all practical purposes, however, we need not make the distinction and can regard $\boldsymbol{\omega}$ as a vector.

3.3 EXPRESSIONS FOR THE MOTION IN TERMS OF MOVING REFERENCE FRAMES

In Sec. 3.2 we considered systems of coordinates rotating relative to an inertial system. In particular, we derived an expression for the rate of change of a vector fixed in the rotating system. Now we wish to obtain an expression for the time derivative of a vector whose components along the moving system vary with time. Refer to Fig. 3.4 and denote by $\mathbf{r}$ the position of point P relative to O when expressed in terms of the components x_i in an inertial space and by $\mathbf{r}'$ when expressed in terms of the components ξ_i along the rotating system. Of course, the two represent the same vector

$$\mathbf{r} = \mathbf{r}' = \xi_1\mathbf{i}' + \xi_2\mathbf{j}' + \xi_3\mathbf{k}'. \tag{3.40}$$

We pointed out in Sec. 3.2 that it is often advantageous to express the motion

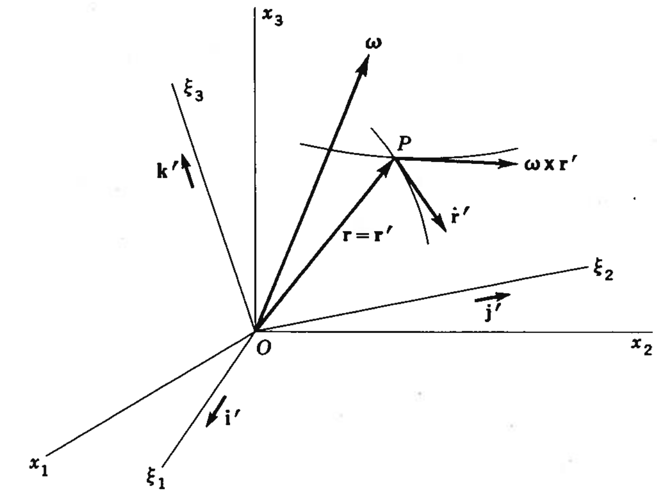


FIGURE 3.4

in terms of components along the rotating system. Differentiating (3.40) with respect to time, we obtain

$$\dot{\mathbf{r}} = \frac{d\xi_1}{dt}\mathbf{i}' + \frac{d\xi_2}{dt}\mathbf{j}' + \frac{d\xi_3}{dt}\mathbf{k}' + \xi_1\frac{d\mathbf{i}'}{dt} + \xi_2\frac{d\mathbf{j}'}{dt} + \xi_3\frac{d\mathbf{k}'}{dt}$$
$$= \dot{\mathbf{r}}' + \boldsymbol{\omega} \times \mathbf{r}', \tag{3.41}$$

where $\quad \dot{\mathbf{r}}' = \dot{\xi}_1\mathbf{i}' + \dot{\xi}_2\mathbf{j}' + \dot{\xi}_3\mathbf{k}' \qquad (3.42)$

denotes the rate of change of $\mathbf{r}'$ relative to the system ξ_1, ξ_2, ξ_3 and $\boldsymbol{\omega} \times \mathbf{r}'$ denotes the rate of change of $\mathbf{r}'$ due to the rotational motion of the system ξ_1, ξ_2, ξ_3. Notice that the latter is precisely the expression derived in Sec. 3.2. In terms of velocities, $\dot{\mathbf{r}}'$ is the velocity of point P relative to the rotating system, and $\boldsymbol{\omega} \times \mathbf{r}'$ is the velocity of the coincident point (a point coinciding with P instantaneously).

Equation (3.41) represents the time derivative of the vector $\mathbf{r}$ in an inertial space when the vector is expressed in terms of a rotating frame of reference and *is valid for any vector*, such as the velocity or the angular momentum vector. Under these circumstances, the second derivative of $\mathbf{r}$ assumes the form

$$\ddot{\mathbf{r}} = \frac{d}{dt}(\dot{\mathbf{r}}') + \frac{d\boldsymbol{\omega}}{dt} \times \mathbf{r}' + \boldsymbol{\omega} \times \frac{d}{dt}(\mathbf{r}')$$
$$= \ddot{\mathbf{r}}' + \boldsymbol{\omega} \times \dot{\mathbf{r}}' + \dot{\boldsymbol{\omega}}' \times \mathbf{r}' + (\boldsymbol{\omega} \times \boldsymbol{\omega}) \times \mathbf{r}' + \boldsymbol{\omega} \times \dot{\mathbf{r}}' + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}')$$
$$= \ddot{\mathbf{r}}' + 2\boldsymbol{\omega} \times \dot{\mathbf{r}}' + \dot{\boldsymbol{\omega}} \times \mathbf{r}' + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}'), \tag{3.43}$$

where $\quad \ddot{\mathbf{r}}' = \ddot{\xi}_1\mathbf{i}' + \ddot{\xi}_2\mathbf{j}' + \ddot{\xi}_3\mathbf{k}' \qquad (3.44)$

is the second derivative of $\mathbf{r}'$ *relative to the rotating system.* In terms of accelerations, $\ddot{\mathbf{r}}$ is the acceleration of P in an inertial space, $\ddot{\mathbf{r}}'$ is the acceleration of P relative to the rotating frame, $2\boldsymbol{\omega} \times \dot{\mathbf{r}}'$ is known as the *Coriolis acceleration*, and $\dot{\boldsymbol{\omega}} \times \mathbf{r}' + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}')$ is the acceleration of the coincident point. The term $\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}')$ is called the *centripetal acceleration* and is directed toward the instantaneous axis of rotation.

If the origin O translates with velocity $\mathbf{v}_0$ and acceleration $\mathbf{a}_0$ with respect to the inertial space, the absolute velocity and acceleration of point P are

$$\mathbf{v} = \mathbf{v}_0 + \dot{\mathbf{r}} \tag{3.45}$$

and

$$\mathbf{a} = \mathbf{a}_0 + \ddot{\mathbf{r}}, \tag{3.46}$$

respectively, where $\dot{\mathbf{r}}$ is given by Eq. (3.41) and $\ddot{\mathbf{r}}$ by Eq. (3.43).

3.4 MOTION RELATIVE TO THE ROTATING EARTH

Although Newton's second law has an extremely simple form when the motion is referred to an inertial system, it frequently is more convenient to refer the motion to a noninertial system. It is natural to assume that there are advantages in referring motion in the vicinity of a point on the earth's surface to a coordinate system rigidly attached to that surface. This indeed proves to be the case. Such a reference frame, however, is not inertial because the earth's center revolves around the sun and the earth rotates about its own axis. The expressions developed in Sec. 3.3 in connection with noninertial reference frames are extremely useful in treating these types of problems.

Although the center of the earth is moving around the sun, the acceleration is relatively small compared to the acceleration due to gravity or even the acceleration of a point on the earth's surface due to the earth's spin, where the point in question is reasonably far from the poles. Furthermore, for the present purpose, the earth's axis of rotation can be assumed to be fixed in space (for a discussion of this assumption, see Sec. 11.8). Hence, we can choose as an inertial system a rectangular set X, Y, Z with the origin at the earth's center C and axis Z aligned with the earth's axis of rotation. Axes X and Y are in the equatorial plane, with X axis pointing toward the vernal equinox (see Fig. 3.5). The earth rotates with an angular velocity Ω, which, for all practical purposes, can be assumed constant and equal to one rotation per day.

We shall be concerned with the motion of a particle in the neighborhood of the earth's surface, the earth being assumed to be a perfect sphere. To express the motion of the particle relative to the earth, we attach a coordinate

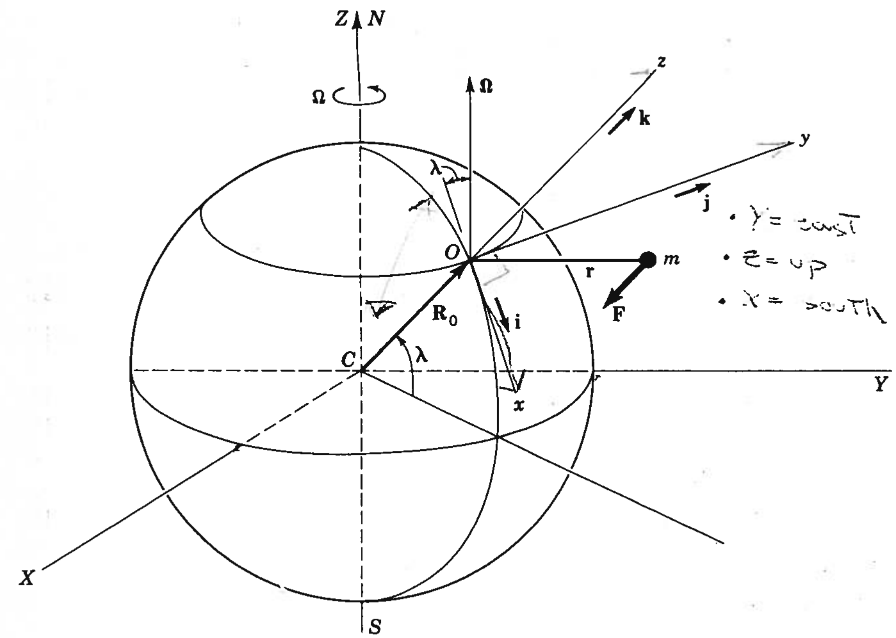


FIGURE 3.5

system x, y, z to the surface of the earth with the origin O situated at a given latitude λ; the longitude turns out to be immaterial. The x axis is tangent to the meridian circle pointing south, y is tangent to the parallel pointing east, and z, since it is directed toward the zenith, coincides with the local vertical. The corresponding unit vectors are $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$. The system x, y, z, being rigidly attached to the earth's surface, possesses the same angular velocity as the earth. From Fig. 3.5 we conclude that the angular velocity can be expressed in terms of components along the x, y, z axes in the following form

$$\boldsymbol{\Omega} = -(\Omega \cos \lambda)\mathbf{i} + (\Omega \sin \lambda)\mathbf{k} = \text{const.} \tag{3.47}$$

The position of the mass m relative to the system x, y, z is denoted by $\mathbf{r}$, and the radius vector from center C to the origin O is denoted by $\mathbf{R}_0$. Hence, using Eq. (3.46), the acceleration of m in an inertial space is

$$\begin{aligned}\mathbf{a} &= \mathbf{a}_0 + \ddot{\mathbf{r}}' + 2\boldsymbol{\omega} \times \dot{\mathbf{r}}' + \dot{\boldsymbol{\omega}}' \times \mathbf{r}' + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') \\ &= \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{R}_0) + \mathbf{a}' + 2\boldsymbol{\Omega} \times \mathbf{v}' + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}),\end{aligned} \tag{3.48}$$

where $\mathbf{v}'$ and $\mathbf{a}'$ are, respectively, the velocity and acceleration of m relative to the x, y, z system.

Let the forces acting upon m be gravity and forces from yet unspecified sources, where the latter are combined into a resultant **F**. For motion in the vicinity of O, the effect of the earth's curvature may be neglected and the gravitational field assumed uniform. It follows that the gravity force vector is constant in magnitude, always parallel to the z axis, and pointing in the negative direction of the z axis. Under these circumstances, Newton's second law takes the form

$$\mathbf{F} - mg\mathbf{k} = m\mathbf{a} = m[\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{R}_0) + \mathbf{a}' + 2\boldsymbol{\Omega} \times \mathbf{v}' + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r})]. \quad (3.49)$$

The last term on the right side of (3.49) is generally very small, so that the differential equations of motion become

$$\begin{aligned} \frac{F_x}{m} &= \ddot{x} - 2\Omega\dot{y}\sin\lambda - \Omega^2 R_0 \sin\lambda\cos\lambda, \\ \frac{F_y}{m} &= \ddot{y} + 2\Omega(\dot{x}\sin\lambda + \dot{z}\cos\lambda), \\ -g + \frac{F_z}{m} &= \ddot{z} - 2\Omega\dot{y}\cos\lambda - \Omega^2 R_0 \cos^2\lambda, \end{aligned} \quad (3.50)$$

which can be solved for the relative motion x, y, z.

3.5 MOTION OF A FREE PARTICLE RELATIVE TO THE ROTATING EARTH

The problem of a particle moving freely relative to the rotating earth has interesting implications. For simplicity let us neglect air resistance and any other forces except gravity, $F_x = F_y = F_z = 0$. Because the rotation of the earth is very small, $\Omega = 7.29 \times 10^{-5}$ rad/sec, second-order terms in Ω lead to accelerations which are negligible compared with the acceleration due to gravity and will be ignored throughout. Under these assumptions, Eqs. (3.50) reduce to

$$\begin{aligned} \ddot{x} - 2\Omega\dot{y}\sin\lambda &= 0, \\ \ddot{y} + 2\Omega(\dot{x}\sin\lambda + \dot{z}\cos\lambda) &= 0, \\ \ddot{z} - 2\Omega\dot{y}\cos\lambda + g &= 0. \end{aligned} \quad (3.51)$$

Assuming that at $t = 0$ the relative position and velocity components are

$$\begin{aligned} x(0) &= 0, \quad & y(0) &= 0, \quad & z(0) &= h, \\ \dot{x}(0) &= u_0, & \dot{y}(0) &= v_0, & \dot{z}(0) &= w_0, \end{aligned} \quad (3.52)$$

we can integrate Eqs. (3.51) once and obtain

$$\begin{aligned} \dot{x} - 2\Omega y\sin\lambda &= u_0, \\ \dot{y} + 2\Omega(x\sin\lambda + z\cos\lambda) &= v_0 + 2\Omega h\cos\lambda, \\ \dot{z} - 2\Omega y\cos\lambda + gt &= w_0. \end{aligned} \quad (3.53)$$

Introducing the first and third of Eqs. (3.53) into the second of Eqs. (3.51) and neglecting terms in Ω^2, we obtain

$$\ddot{y} + 2\Omega(u_0\sin\lambda + w_0\cos\lambda) - 2\Omega gt\cos\lambda = 0, \quad (3.54)$$

which can be integrated twice with the result

$$y = v_0 t - \Omega t^2(u_0\sin\lambda + w_0\cos\lambda) + \tfrac{1}{3}\Omega g t^3\cos\lambda. \quad (3.55)$$

A substitution of Eq. (3.55) into the first and third of Eqs. (3.53) and integrations with respect to time yield

$$\begin{aligned} x &= u_0 t + \Omega v_0 t^2 \sin\lambda, \\ z &= h + w_0 t + \Omega v_0 t^2\cos\lambda - \tfrac{1}{2}gt^2, \end{aligned} \quad (3.56)$$

where again terms in Ω^2 have been ignored.

If a particle is dropped from rest, $u_0 = v_0 = w_0 = 0$, at a height h, it will land on the earth's surface after an interval of time

$$t = \sqrt{\frac{2h}{g}}. \quad (3.57)$$

The coordinates of the landing point are

$$x = z = 0, \qquad y = \tfrac{2}{3}\Omega h\left(\frac{2h}{g}\right)^{\frac{1}{2}}\cos\lambda, \quad (3.58)$$

where the positive sign of y indicates that it lands at a point on the y axis *east of the origin*. This result may seem a little puzzling in view of the fact that the earth is rotating from west to east. This can be easily explained, however, since as the particle drops with a downward velocity gt, it is being deflected eastward by the Coriolis effect. Due to the nature of the assumptions, the above result is difficult to verify experimentally.

Another case of interest which can be attributed directly to the Coriolis effect but is easier to detect is the cyclone, created when a point of low pressure is surrounded by points of high pressure. The winds rushing toward the point of low pressure are deflected so as to generate a cyclone. In Fig. 3.6, which shows an idealized situation as it might occur in the Northern Hemisphere, the concentric circles represent isobars, with the pressure decreasing toward the center. The dashed arrows represent the direction the winds would have in the absence of the earth's rotation, whereas the solid arrows include the deflection produced by the Coriolis effect, as reflected by the

It is often convenient to choose as the six coordinates describing the motion of a rigid body three translations of a certain point within the body and three rotations about that point. To this end, a system of axes, called *body axes*, is embedded in the body, and the motion is described in terms of the translation of the origin O of the body axes as well as the rotation of these axes with respect to an inertial space. But (by design and not by coincidence) this is just how the motion was described in Chap. 3, so that the derivations there can be applied equally well here. In fact, the description of the motion of a rigid body is simpler because there is no motion of the mass points with respect to the body axes. Hence, from Eqs. (3.41), (3.43), (3.45), and (3.46) we obtain the absolute velocity of a point in a rigid body

$$\mathbf{v} = \mathbf{v}_0 + \boldsymbol{\omega} \times \mathbf{r} \tag{4.1}$$

and the absolute acceleration

$$\mathbf{a} = \mathbf{a}_0 + \dot{\boldsymbol{\omega}} \times \mathbf{r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) \tag{4.2}$$

after setting $\dot{\mathbf{r}}' = \ddot{\mathbf{r}}' = \mathbf{0}$ and letting $\mathbf{r} = \mathbf{r}'$ be the radius vector of the point in question measured with respect to the body axes.

The case in which one of the points of the body is fixed in an inertial space is of considerable interest in rigid body dynamics. The orientation of the body can be defined in terms of an orthogonal transformation given by the 3×3 matrix $[l(t)]$ relating the body axes to the inertial system at any time t. If initially the body axes were coincident with the inertial system, then $[l(0)] = [1]$, where the latter is the unit matrix.

Next we shall show that the general displacement of a rigid body with one point fixed is a rotation about some axis through the fixed point. This implies that in the inertial space there is an axis in the rigid body left unaffected by the rotation, which is equivalent to saying that the components of a vector coinciding with the axis of rotation remain the same as before the rotation. Using Eq. (3.4), this can be expressed mathematically in the matrix form

$$\{\xi\} = [l]\{x\} = \{x\}. \tag{4.3}$$

Now let us consider the general *eigenvalue problem*

$$[l]\{x\} = \lambda\{x\}, \tag{4.4}$$

of which Eq. (4.3) is a special case. The homogeneous equation (4.4) has nontrivial solutions for only certain values of λ. These constants, called *eigenvalues*, are denoted by λ_r, and the corresponding vectors, known as *eigenvectors*, are denoted by $\{x^{(r)}\}$ $(r = 1, 2, 3)$ (see Ref. 2, sec. 4-3). Thus the real orthogonal matrix $[l]$, representing the rotation of a rigid body, must have an eigenvalue equal to $+1$ if the statement concerning the general displacement of a rigid body is to be true.

In general, Eq. (4.4) has three distinct eigenvalues λ_r, which can be arranged in a diagonal matrix $[\lambda]$, and three associated eigenvectors $\{x^{(r)}\}$, forming the square matrix $[x]$, so that Eq. (4.4) can be rewritten

$$[l][x] = [x][\lambda]. \tag{4.5}$$

Premultiplying Eq. (4.5) through by $[x]^{-1}$, we obtain

$$[x]^{-1}[l][x] = [\lambda], \tag{4.6}$$

so that the solution of the eigenvalue problem reduces to finding a matrix $[x]$ which transforms $[l]$ into a diagonal matrix. A transformation of the type (4.6) is known as a *similarity transformation*. Certain properties of similarity transformations will be introduced as needed.

Because $[l]$ is not symmetric, although it is real, some of the eigenvalues may be complex, from which it follows that the associated eigenvectors must also be complex. Complex eigenvectors have no meaning as far as our physical problem is concerned, but this point turns out to be irrelevant. What is relevant is that we must insist that the orthogonal transformation in question does not affect the magnitude of the vector $\{x\}$, so that denoting by $\{\xi^*\}$ and $\{x^*\}$ the complex conjugates of $\{\xi\}$ and $\{x\}$, we must have

$$\{\xi^*\}^T\{\xi\} = \{x^*\}^T\{x\}, \tag{4.7}$$

and if $\{x\}$ is an eigenvector, we must also have

$$\{x^*\}^T\{x\} = \lambda^*\lambda\{x^*\}^T\{x\}, \tag{4.8}$$

where λ^* is the complex conjugate of the eigenvalue λ. From Eq. (4.8) it follows that

$$\lambda^*\lambda = 1, \tag{4.9}$$

or the magnitude of the eigenvalues is equal to unity.

The matrices $[l]$ and $[\lambda]$ of Eq. (4.6) are said to be similar, which implies that their eigenvalues are equal and so are the values of the corresponding characteristic determinants. This allows us to write

$$|l_{ij} - \lambda\delta_{ij}| = |(\lambda_{ij} - \lambda)\delta_{ij}| = 0, \tag{4.10}$$

where δ_{ij} is the Kronecker delta and the notation $\lambda_i = \lambda_{ii}$ $(i = 1, 2, 3)$ has been used. Expanding the two determinants, we conclude that

$$|l| = \lambda_1\lambda_2\lambda_3, \tag{4.11}$$

where, because all the elements l_{ij} are real, the determinant $|l|$ is real. It follows that at least one of the eigenvalues must be real and the other two are either real or complex conjugates.

In Sec. 3.1 we showed that $|l| = \pm 1$, so that the product of the three eigenvalues must equal ± 1. But the value $|l| = -1$ corresponds to an