Problems for Sec. 8.2

Find $\mathbf{u}'$, $|\mathbf{u}'|$, $\mathbf{u}''$, and $|\mathbf{u}''|$, where $\mathbf{u}$ equals

1. $\mathbf{a} + \mathbf{b}t$
2. $t\mathbf{i} + t^2\mathbf{j}$
3. $\cos t\,\mathbf{i} + \sin t\,\mathbf{j}$
4. $\cos t\,\mathbf{i} + 4\sin t\,\mathbf{j} + \mathbf{k}$
5. $t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$
6. $2\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j} + t\,\mathbf{k}$
7. $e^t\mathbf{i} + e^{-t}\mathbf{j}$
8. $e^{-t}(\cos t\,\mathbf{i} + \sin t\,\mathbf{j})$
9. $\sin 3t\,(\mathbf{i} + \mathbf{j})$

Let $\mathbf{u} = t\mathbf{i} + 2t^2\mathbf{k}$, $\mathbf{v} = t^3\mathbf{j} + t\mathbf{k}$, and $\mathbf{w} = \mathbf{i} + t\mathbf{j} + t^2\mathbf{k}$. Find

10. $(\mathbf{u} \cdot \mathbf{v})'$
11. $(\mathbf{u} \times \mathbf{v})'$
12. $(\mathbf{u}\ \mathbf{v}\ \mathbf{w})'$
13. $[(\mathbf{u} \times \mathbf{v}) \times \mathbf{w}]'$
14. $[\mathbf{u} \times (\mathbf{v} \times \mathbf{w})]'$
15. $[(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w}]'$

In each case find the first partial derivatives with respect to x, y, z.

16. $x\mathbf{i} + 2y\mathbf{j}$
17. $(x^2 - y^2)\mathbf{i} + 2xy\mathbf{j}$
18. $x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$
19. $yz\mathbf{i} + zx\mathbf{j} + xy\mathbf{k}$
20. $(x + y)\mathbf{j} + (x - y)\mathbf{k}$
21. $x^2y\mathbf{i} + y^2z\mathbf{j} + z^2x\mathbf{k}$

22. Using (8) in Sec. 6.5, prove (8). Prove (9).
23. Derive (10) from (8) and (9).
24. Give a direct proof of (10).
25. Find formulas similar to (8) and (9) for $(\mathbf{u} \cdot \mathbf{v})''$ and $(\mathbf{u} \times \mathbf{v})''$.
26. Show that if $\mathbf{u}(t)$ is a unit vector and $\mathbf{u}'(t) \neq \mathbf{0}$, then $\mathbf{u}$ and $\mathbf{u}'$ are orthogonal.
27. Show that the equation $\mathbf{u}'(t) = \mathbf{c}$ has the solution $\mathbf{u}(t) = \mathbf{c}t + \mathbf{b}$ where $\mathbf{c}$ and $\mathbf{b}$ are constant vectors.
28. Show that $\mathbf{u}(t) = \mathbf{b}e^{\lambda t} + \mathbf{c}e^{-\lambda t}$ satisfies the equation $\mathbf{u}'' - \lambda^2\mathbf{u} = \mathbf{0}$. ($\mathbf{b}$ and $\mathbf{c}$ are constant vectors.)
29. Show that $\left(\dfrac{\mathbf{u}}{|\mathbf{u}|}\right)' = \dfrac{\mathbf{u}'(\mathbf{u} \cdot \mathbf{u}) - \mathbf{u}(\mathbf{u} \cdot \mathbf{u}')}{(\mathbf{u} \cdot \mathbf{u})^{3/2}}$.
30. Differentiate $(t\mathbf{i} + t^2\mathbf{j})/|t\mathbf{i} + t^2\mathbf{j}|$.

8.3 Curves

As an important application of vector calculus, let us now consider some basic facts about curves in space. The student will know that curves occur in many considerations in calculus as well as in physics, for example, as paths of moving particles. The consideration will be a part of an important branch of mathematics, which is called **differential geometry** and which may be defined as the study of curves and surfaces by means of calculus. Cf. Ref. [C8] in Appendix 1.

A Cartesian coordinate system being given, we may represent a curve C by a vector function (Fig. 151 on the next page)

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}; \tag{1}$$

to each value t_0 of the real variable t there corresponds a point of C having the position vector $\mathbf{r}(t_0)$, that is, the coordinates $x(t_0)$, $y(t_0)$, $z(t_0)$.

A representation of the form (1) is called a **parametric representation** of the curve C, and t is called the *parameter* of this representation. This type of representation is useful in many applications, for example, in mechanics where the variable t may be time.

Other types of representations of curves in space are

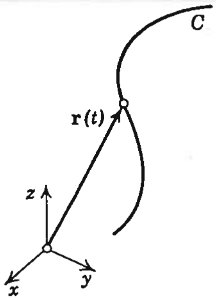

Fig. 151. Parametric representation of a curve

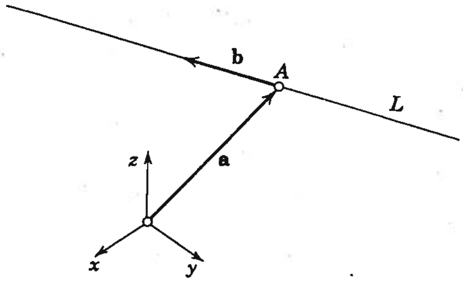

Fig. 152. Parametric representation of a straight line

(2) $$y = f(x), \qquad z = g(x)$$

and

(3) $$F(x, y, z) = 0, \qquad G(x, y, z) = 0.$$

By setting $x = t$ we may write (2) in the form (1), namely

$$\mathbf{r}(t) = t\mathbf{i} + f(t)\mathbf{j} + g(t)\mathbf{k}.$$

In (3), each equation represents a surface, and the curve is the intersection of the two surfaces.

A **plane curve** is a curve which lies in a plane in space. A curve which is not a plane curve is called a **twisted curve.**

Example 1. Straight line

Any straight line L can be represented in the form

(4) $$\mathbf{r}(t) = \mathbf{a} + t\mathbf{b} = (a_1 + tb_1)\mathbf{i} + (a_2 + tb_2)\mathbf{j} + (a_3 + tb_3)\mathbf{k}$$

where $\mathbf{a}$ and $\mathbf{b}$ are constant vectors. L passes through the point A with position vector $\mathbf{r} = \mathbf{a}$ and has the direction of $\mathbf{b}$ (Fig. 152). If $\mathbf{b}$ is a unit vector, its components are the *direction cosines* of L, and in this case, $|t|$ measures the distance of the points of L from A.

Example 2. Ellipse, circle

The vector function

(5) $$\mathbf{r}(t) = a \cos t\,\mathbf{i} + b \sin t\,\mathbf{j}$$

represents an ellipse in the xy-plane with center at the origin and principal axes in the direction of the x and y axes. In fact, since $\cos^2 t + \sin^2 t = 1$, we obtain from (5)

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \qquad z = 0.$$

If $b = a$, then (5) represents a *circle* of radius a.

Example 3. Circular helix

The twisted curve C represented by the vector function

(6) $$\mathbf{r}(t) = a \cos t\,\mathbf{i} + a \sin t\,\mathbf{j} + ct\mathbf{k} \qquad (c \neq 0)$$

is called a *circular helix*. It lies on the cylinder $x^2 + y^2 = a^2$. If $c > 0$, the helix is shaped like a right-handed screw (Fig. 153). If $c < 0$, it looks like a left-handed screw (Fig. 154). ∎

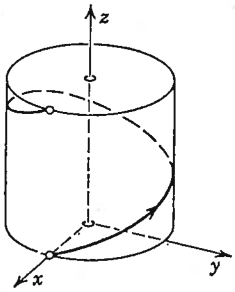

Fig. 153. Right-handed circular helix

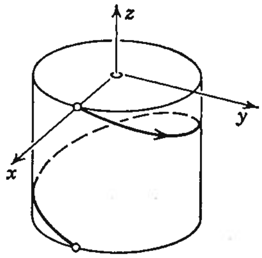

Fig. 154. Left-handed circular helix

The portion between any two points of a curve is often called an **arc** *of a curve*. For the sake of simplicity we shall use the single term "curve" to denote an entire curve as well as an arc of a curve.

A curve may have self-intersections; the points of intersection are called *multiple points* of the curve. Two examples are shown in Fig. 155. A curve having no multiple points is called a **simple curve.**

Example 4. Simple and nonsimple curves

Ellipses and helices are simple curves. The curve represented by

$$\mathbf{r}(t) = (t^2 - 1)\mathbf{i} + (t^3 - t)\mathbf{j}$$

is not simple since it has a double point at the origin; this point corresponds to the two values $t = 1$ and $t = -1$. ∎

We finally mention that a given curve C may be represented by various vector functions. For example, if C is represented by (1) and we set $t = h(t^*)$, then we obtain a new vector function $\tilde{\mathbf{r}}(t^*) = \mathbf{r}(h(t^*))$ representing C, provided $h(t^*)$ takes on all the values of t occurring in (1).

Example 5. Change of parameter

The parabola $y = x^2$ in the xy-plane may be represented by the vector function

$$\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} \qquad (-\infty < t < \infty).$$

If we set $t = -2t^*$, we obtain another representation of the parabola:

$$\tilde{\mathbf{r}}(t^*) = \mathbf{r}(-2t^*) = -2t^*\mathbf{i} + 4t^{*2}\mathbf{j}.$$

If we set $t = t^{*2}$, we obtain

$$\tilde{\mathbf{r}}(t^*) = t^{*2}\mathbf{i} + t^{*4}\mathbf{j},$$

but this function represents only the portion of the parabola in the first quadrant, because $t^{*2} \geqq 0$ for all t^*.

Fig. 155. Curves having double points

Problems for Sec. 8.3

Find a parametric representation of the straight line through a point A in the direction of a vector $\mathbf{b}$, where

1. A: $(0, 0, 0)$, $\mathbf{b} = \mathbf{i} + \mathbf{j}$
2. A: $(1, 3, 2)$, $\mathbf{b} = -\mathbf{i} + \mathbf{k}$
3. A: $(2, 1, 0)$, $\mathbf{b} = 2\mathbf{j} + \mathbf{k}$
4. A: $(0, 4, 1)$, $\mathbf{b} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$

Find a parametric representation of the straight line through the points A and B, where

5. A: $(0, 0, 0)$, B: $(1, 1, 1)$
6. A: $(-1, 8, 3)$, B: $(1, 0, 0)$
7. A: $(1, 5, 3)$, B: $(0, 2, -1)$
8. A: $(1, 4, 2)$, B: $(1, 4, -2)$

Find a parametric representation of the straight line represented by

9. $y = x, \quad z = 0$
10. $7x - 3y + z = 14, \quad 4x - 3y - 2z = -1$
11. $x + y = 0, \quad x - z = 0$
12. $x + y + z = 1, \quad y - z = 0$

Represent the following curves in parametric form and sketch these curves.

13. $x^2 + y^2 = 1, \quad z = 0$
14. $y = x^4, \quad z = 0$
15. $y = x^2, \quad z = x^3$
16. $x^2 + y^2 - 2x - 4y = -1, \quad z = 0$
17. $4(x + 1)^2 + y^2 = 4, \quad z = 0$
18. $x^2 + y^2 = 4, \quad z = e^x$

19. Determine the orthogonal projections of the circular helix (6) in the coordinate planes.

Sketch figures of the curves represented by the following functions $\mathbf{r}(t)$.

20. $t\mathbf{i} + 2t\mathbf{j} - t\mathbf{k}$
21. $2 \sin t\, \mathbf{i} + 2 \cos t\, \mathbf{j}$
22. $(1 + \cos t)\mathbf{i} + \sin t\, \mathbf{j}$
23. $\cos t\, \mathbf{i} + 2 \sin t\, \mathbf{j}$
24. $\cos t\, \mathbf{i} + \sin t\, \mathbf{j} + t\, \mathbf{k}$
25. $t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$

8.4 Arc Length

To define the length of a simple curve C we may proceed as follows. We inscribe in C a broken line of n chords joining the two endpoints of C as shown in Fig. 156. This we do for each positive integer n in an arbitrary way but such that the maximum chord-length approaches zero as n approaches infinity. The lengths of these lines of chords can be obtained from the theorem of Pythagoras. If the sequence of these lengths $l_1, l_2, \cdots$ is convergent with limit l, then C is said to be **rectifiable**, and l is called the **length** of C.

If C is not simple but consists of finitely many rectifiable simple curves, the

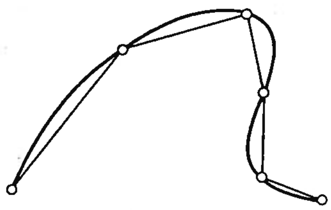

Fig. 156. Length of a curve

length of C is defined to be the sum of the lengths of those curves.

If C can be represented by a continuously differentiable[2] vector function

$$\mathbf{r} = \mathbf{r}(t) \qquad (a \leqq t \leqq b),$$

then it can be shown that C is rectifiable, and its length l is given by the integral

$$\text{(1)} \qquad l = \int_a^b \sqrt{\dot{\mathbf{r}} \cdot \dot{\mathbf{r}}}\, dt \qquad \left(\dot{\mathbf{r}} = \frac{d\mathbf{r}}{dt}\right),$$

whose value is independent of the choice of the parametric representation. The proof is quite similar to that for plane curves usually considered in elementary integral calculus (cf. Ref. [A14]) and can be found in Ref. [C8] in Appendix 1.

If we replace the fixed upper limit b in (1) with a variable upper limit t, the integral becomes a function of t, say, $s(t)$; denoting the variable of integration by t^*, we have

$$\text{(2)} \qquad s(t) = \int_a^t \sqrt{\dot{\mathbf{r}} \cdot \dot{\mathbf{r}}}\, dt^* \qquad \left(\dot{\mathbf{r}} = \frac{d\mathbf{r}}{dt^*}\right).$$

This function $s(t)$ is called the *arc length function* or, simply, the **arc length** of C.

From our consideration it follows that, geometrically, for a fixed $t = t_0 \geqq a$, the arc length $s(t_0)$ is the length of the portion of C between the points corresponding to $t = a$ and $t = t_0$. For $t = t_0 < a$ we have $s(t_0) < 0$ and that length is $-s(t_0)$.

The arc length s may serve as a parameter in parametric representations of curves. We shall see that this leads to a simplification of various formulas.

The constant a in (2) may be replaced by another constant; that is, the point of the curve corresponding to $s = 0$ may be chosen in an arbitrary manner. The sense corresponding to increasing values of s is called the **positive sense** on C; in this fashion any representation $\mathbf{r}(s)$ or $\mathbf{r}(t)$ of C defines a certain **orientation** of C. Obviously, there are two ways of *orienting* C, and it is not difficult to see that the transition from one orientation to the opposite orientation can be effected by a transformation of the parameter whose derivative is negative.

From (2) we obtain

$$\text{(3)} \qquad \left(\frac{ds}{dt}\right)^2 = \frac{d\mathbf{r}}{dt} \cdot \frac{d\mathbf{r}}{dt} = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2.$$

It is customary to write

$$d\mathbf{r} = dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k}$$

and

$$\text{(4)} \qquad ds^2 = d\mathbf{r} \cdot d\mathbf{r} = dx^2 + dy^2 + dz^2.$$

ds is called the **linear element** of C.

[2] "*Continuously differentiable*" means that the derivative exists and is continuous; "*twice continuously differentiable*" means that the first and second derivatives exist and are continuous, and so on.

Example 1. Circle. Arc length as parameter
In the case of the circle

$$\mathbf{r}(t) = a \cos t\, \mathbf{i} + a \sin t\, \mathbf{j}$$

we have $\dot{\mathbf{r}} = -a \sin t\, \mathbf{i} + a \cos t\, \mathbf{j}$, $\dot{\mathbf{r}} \cdot \dot{\mathbf{r}} = a^2$, and therefore,

$$s(t) = \int_0^t a\, dt^* = at.$$

Hence $t(s) = s/a$ and a representation of the circle with the arc length s as parameter is

$$\mathbf{r}\left(\frac{s}{a}\right) = a \cos \frac{s}{a}\, \mathbf{i} + a \sin \frac{s}{a}\, \mathbf{j}.$$

The circle is oriented in the counterclockwise sense, which corresponds to increasing values of s. Setting $s = -\tilde{s}$ and remembering that $\cos(-\alpha) = \cos \alpha$ and $\sin(-\alpha) = -\sin \alpha$, we obtain

$$\mathbf{r}\left(-\frac{\tilde{s}}{a}\right) = a \cos \frac{\tilde{s}}{a}\, \mathbf{i} - a \sin \frac{\tilde{s}}{a}\, \mathbf{j};$$

we have $ds/d\tilde{s} = -1 < 0$, and the circle is now oriented in the clockwise sense.

Problems for Sec. 8.4

Graph the following curves and find their lengths.

1. *Catenary* $y = \cosh x$, $z = 0$, from $x = 0$ to $x = 1$
2. *Circular helix* $\mathbf{r}(t) = a \cos t\, \mathbf{i} + a \sin t\, \mathbf{j} + ct\mathbf{k}$ from $(a, 0, 0)$ to $(a, 0, 2\pi c)$
3. *Semicubical parabola* $y = x^{3/2}$, $z = 0$, from $(0, 0, 0)$ to $(4, 8, 0)$
4. Four-cusped *hypocycloid* $\mathbf{r}(t) = a \cos^3 t\, \mathbf{i} + a \sin^3 t\, \mathbf{j}$, total length
5. *Involute of circle* $\mathbf{r}(t) = (\cos t + t \sin t)\mathbf{i} + (\sin t - t \cos t)\mathbf{j}$ from $(1, 0, 0)$ to $(-1, \pi, 0)$
6. $\mathbf{r}(t) = e^t \cos t\, \mathbf{i} + e^t \sin t\, \mathbf{j}$, $0 \leqq t \leqq \pi/2$

7. If a plane curve is represented in the form $y = f(x)$, $z = 0$, using (1) show that its length between $x = a$ and $x = b$ is

$$l = \int_a^b \sqrt{1 + y'^2}\, dx.$$

8. Using the formula in Prob. 7, find the length of a circle of radius a.
9. Show that if a plane curve is represented in polar coordinates $\rho = \sqrt{x^2 + y^2}$ and $\theta = \arctan(y/x)$, then $ds^2 = \rho^2\, d\theta^2 + d\rho^2$.

Using the formula in Prob. 9, find the lengths of the following curves.

10. Circle of radius a, total length
11. $\rho = e^\theta$, $0 \leqq \theta \leqq \pi$
12. $\rho = \theta^2$, $0 \leqq \theta \leqq \pi/2$
13. *Cardioid* $\rho = a(1 - \cos \theta)$. (Graph this curve.)
14. $\rho = 1 + \cos \theta$, $0 \leqq \theta \leqq \pi/2$

15. If a curve is represented by a parametric representation, show that a transformation of the parameter whose derivative is negative reverses the orientation.

8.5 Tangent. Curvature and Torsion

The *tangent* to a curve C at a point P of C is defined as the limiting position of the straight line L through P and another point Q of C as Q approaches P along the curve (Fig. 157).

Suppose that C is represented by a continuously differentiable vector function $\mathbf{r}(t)$ where t is any parameter. Let P and Q correspond to t and $t + \Delta t$, respectively. Then L has the direction of the vector

$$[\mathbf{r}(t + \Delta t) - \mathbf{r}(t)]/\Delta t.$$

Hence, if the vector

$$\dot{\mathbf{r}} = \lim_{\Delta t \to 0} \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t} \tag{1}$$

is not the zero vector, it has the direction of the tangent to C at P. It points in the direction of increasing values of t, and its sense, therefore, depends on the orientation of the curve. $\dot{\mathbf{r}}$ is called a *tangent vector* of C at P, and the corresponding unit vector

$$\mathbf{u} = \frac{\dot{\mathbf{r}}}{|\dot{\mathbf{r}}|} \tag{2}$$

is called the **unit tangent vector** to C at P.

If in particular C is represented by $\mathbf{r}(s)$, where s is the arc length, it follows from (3), Sec. 8.4, that the derivative $d\mathbf{r}/ds$ is a unit vector, and (2) becomes

$$\mathbf{u} = \mathbf{r}' = \frac{d\mathbf{r}}{ds}. \tag{3}$$

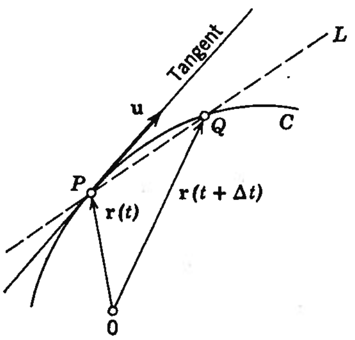

Fig. 157. Tangent to a curve

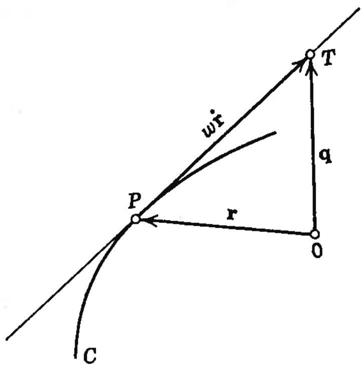

Fig. 158. Representation of the tangent to a curve

Clearly the position vector of a point T on the tangent is the sum of the position vector $\mathbf{r}$ of P and a vector in the direction of the tangent. Hence a parametric representation of the tangent is (Fig. 158)

$$\mathbf{q}(w) = \mathbf{r} + w\dot{\mathbf{r}} \tag{4}$$

where both $\mathbf{r}$ and $\dot{\mathbf{r}}$ depend on P and the parameter w is a real variable.

Consider a given curve C, represented by a three times continuously differentiable vector function $\mathbf{r}(s)$ (cf. footnote 2 in Sec. 8.4) where s is the arc length. Then

$$\kappa(s) = |\mathbf{u}'(s)| = |\mathbf{r}''(s)| \qquad (\kappa \geqq 0) \tag{5}$$

is called the **curvature** of C. If $\kappa \neq 0$, the unit vector $\mathbf{p}$ in the direction of $\mathbf{u}'(s)$ is

$$\mathbf{p} = \frac{\mathbf{u}'}{\kappa} \qquad (\kappa > 0) \tag{6}$$

and is called the **unit principal normal vector** of C. From Example 1 in Sec. 8.2 we see that $\mathbf{p}$ is perpendicular to $\mathbf{u}$. The vector

$$\mathbf{b} = \mathbf{u} \times \mathbf{p} \qquad (\kappa > 0) \tag{7}$$

is called the **unit binormal vector** of C. From the definition of a vector product it follows that $\mathbf{u}, \mathbf{p}, \mathbf{b}$ constitute a right-handed triple of orthogonal unit vectors (Secs. 6.3 and 6.7). This triple is called the **trihedron** of C at the point under consideration (Fig. 159). The three straight lines through that point in the directions of $\mathbf{u}, \mathbf{p}, \mathbf{b}$ are called the **tangent,** the **principal normal,** and the **binormal** of C. Figure 159 also shows the names of the three planes spanned by each pair of those vectors.

If the derivative $\mathbf{b}'$ is not the zero vector, it is perpendicular to $\mathbf{b}$ (cf. Example 1 in Sec. 8.2). It is also perpendicular to $\mathbf{u}$. In fact, by differentiating $\mathbf{b} \cdot \mathbf{u} = 0$ we have $\mathbf{b}' \cdot \mathbf{u} + \mathbf{b} \cdot \mathbf{u}' = 0$; hence $\mathbf{b}' \cdot \mathbf{u} = 0$ because $\mathbf{b} \cdot \mathbf{u}' = 0$. Consequently, $\mathbf{b}'$ is of the form $\mathbf{b}' = \alpha\mathbf{p}$ where α is a scalar. It is customary to set $\alpha = -\tau$. Then

$$\mathbf{b}' = -\tau\mathbf{p} \qquad (\kappa > 0). \tag{8}$$

The scalar function τ is called the **torsion** of C. Scalar multiplication of both sides of (8) by $\mathbf{p}$ yields

$$\tau(s) = -\mathbf{p}(s) \cdot \mathbf{b}'(s). \tag{9}$$

The concepts just introduced are basic in the theory and application of curves. Let us illustrate them by a typical example. Further applications will follow later.

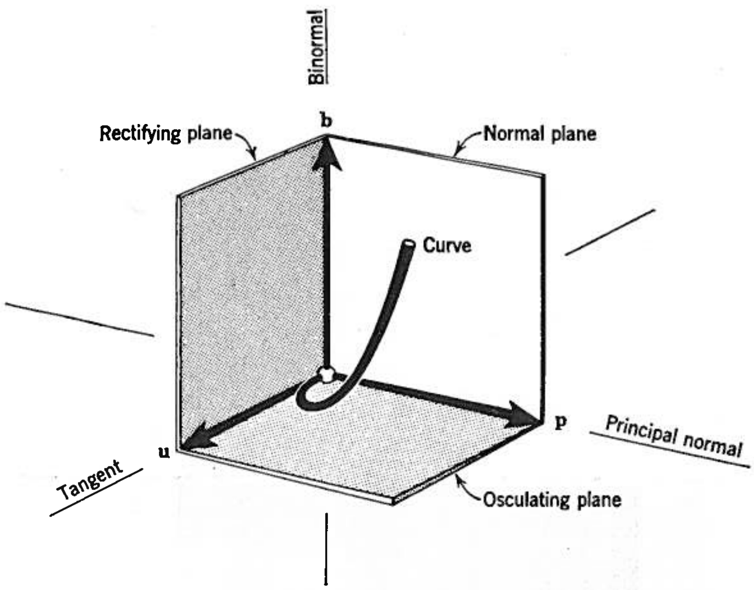

Fig. 159. Trihedron

Example 1. Circular helix

In the case of the circular helix (6) in Sec. 8.3 we obtain the arc length $s = t\sqrt{a^2 + c^2}$. Hence we may represent the helix in the form

$$\mathbf{r}(s) = a \cos\frac{s}{K}\mathbf{i} + a \sin\frac{s}{K}\mathbf{j} + c\frac{s}{K}\mathbf{k} \quad \text{where} \quad K = \sqrt{a^2 + c^2}.$$

It follows that

$$\mathbf{u}(s) = \mathbf{r}'(s) = -\frac{a}{K}\sin\frac{s}{K}\mathbf{i} + \frac{a}{K}\cos\frac{s}{K}\mathbf{j} + \frac{c}{K}\mathbf{k}$$

$$\mathbf{r}''(s) = -\frac{a}{K^2}\cos\frac{s}{K}\mathbf{i} - \frac{a}{K^2}\sin\frac{s}{K}\mathbf{j}$$

$$\kappa = |\mathbf{r}''| = \sqrt{\mathbf{r}''\cdot\mathbf{r}''} = \frac{a}{K^2} = \frac{a}{a^2 + c^2}$$

$$\mathbf{p}(s) = \frac{\mathbf{r}''(s)}{\kappa(s)} = -\cos\frac{s}{K}\mathbf{i} - \sin\frac{s}{K}\mathbf{j}$$

$$\mathbf{b}(s) = \mathbf{u}(s) \times \mathbf{p}(s) = \frac{c}{K}\sin\frac{s}{K}\mathbf{i} - \frac{c}{K}\cos\frac{s}{K}\mathbf{j} + \frac{a}{K}\mathbf{k}$$

$$\mathbf{b}'(s) = \frac{c}{K^2}\cos\frac{s}{K}\mathbf{i} + \frac{c}{K^2}\sin\frac{s}{K}\mathbf{j}$$

$$\tau(s) = -\mathbf{p}(s)\cdot\mathbf{b}'(s) = \frac{c}{K^2} = \frac{c}{a^2 + c^2}.$$

Hence the circular helix has constant curvature and torsion. If $c > 0$ (right-handed helix, cf. Fig. 153), then $\tau > 0$, and if $c < 0$ (left-handed helix, cf. Fig. 154), then $\tau < 0$. ∎

Since **u**, **p**, and **b** are linearly independent vectors, we may represent any vector in space as a linear combination of these vectors. Hence if the derivatives **u**′, **p**′, and **b**′ exist, they may be represented in that fashion. The corresponding formulas are

$$
\text{(10)}\qquad
\begin{aligned}
&\text{(a)}\quad & \mathbf{u}' &= && \kappa\mathbf{p} && \\
&\text{(b)}\quad & \mathbf{p}' &= -\kappa\mathbf{u} && && +\tau\mathbf{b} \\
&\text{(c)}\quad & \mathbf{b}' &= && -\tau\mathbf{p} &&
\end{aligned}
$$

They are called the **formulas of Frenet.** (10a) follows from (6), and (10c) is identical with (8). To derive (10b) we note that, by the definition of a vector product,

$$\mathbf{p} = \mathbf{b} \times \mathbf{u}, \qquad \mathbf{p} \times \mathbf{u} = -\mathbf{b}, \qquad \mathbf{b} \times \mathbf{p} = -\mathbf{u}.$$

Differentiating the first of these formulas and using (10a) and (10c), we obtain

$$\mathbf{p}' = \mathbf{b}' \times \mathbf{u} + \mathbf{b} \times \mathbf{u}' = -\tau\mathbf{p} \times \mathbf{u} + \mathbf{b} \times \kappa\mathbf{p} = -\tau(-\mathbf{b}) + \kappa(-\mathbf{u}),$$

which proves (10b).

Problems for Sec. 8.5

Find a parametric representation of the tangent of the following curves at the given point P.

1. $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j}$, $\quad P\colon (-1/\sqrt{2}, 1/\sqrt{2})$
2. $\mathbf{r}(t) = t\,\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$, $\quad P\colon (1, 1, 1)$
3. $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + 2t\,\mathbf{k}$, $\quad P\colon (1, 0, 4\pi)$
4. $\mathbf{r}(t) = 4\cos t\,\mathbf{i} + 4\sin t\,\mathbf{j}$, $\quad P\colon (2\sqrt{2}, -2\sqrt{2}, 0)$

5. Show that in Example 1, the angle between **u** and the z-axis is constant.
6. Show that straight lines are the only curves whose unit tangent vector is constant.
7. Show that the curvature of a straight line is identically zero.
8. Show that if the curve C is represented by $\mathbf{r}(t)$ where t is any parameter, then the curvature is

$$\text{(5')}\qquad \kappa = \frac{\sqrt{(\dot{\mathbf{r}} \cdot \dot{\mathbf{r}})(\ddot{\mathbf{r}} \cdot \ddot{\mathbf{r}}) - (\dot{\mathbf{r}} \cdot \ddot{\mathbf{r}})^2}}{(\dot{\mathbf{r}} \cdot \dot{\mathbf{r}})^{3/2}}.$$

9. Show that the curvature of a circle of radius a equals $1/a$.
10. Find the curvature of the ellipse $\mathbf{r}(t) = a\cos t\,\mathbf{i} + b\sin t\,\mathbf{j}$.
11. Using (5′), show that for a curve $y = y(x)$ in the xy-plane,

$$\kappa = |y''|/(1 + y'^2)^{3/2} \qquad (y' = dy/dx, \text{ etc.}).$$

12. Show that the torsion of a plane curve (with $\kappa > 0$) is identically zero.
13. Using (7) and (9), show that

$$\text{(9')}\qquad \tau = (\mathbf{u}\ \mathbf{p}\ \mathbf{p}') \qquad (\kappa > 0).$$

14. Using (6), show that (9′) may be written

$$\text{(9'')}\qquad \tau = (\mathbf{r}'\ \mathbf{r}''\ \mathbf{r}''')/\kappa^2 \qquad (\kappa > 0).$$

15. Show that if the curve C is represented by $\mathbf{r}(t)$ where t is any parameter, then (9″) becomes

$$\text{(9''')}\qquad \tau = \frac{(\dot{\mathbf{r}}\ \ddot{\mathbf{r}}\ \dddot{\mathbf{r}})}{(\dot{\mathbf{r}} \cdot \dot{\mathbf{r}})(\ddot{\mathbf{r}} \cdot \ddot{\mathbf{r}}) - (\dot{\mathbf{r}} \cdot \ddot{\mathbf{r}})^2} \qquad (\kappa > 0).$$