APPENDIX A
APPENDIX FOR GRASP ANALYSIS

A.1 Matrix Identities

The finger positions and orientations may be expressed with 4x4 homogeneous transformation matrices, [T]:

$$[T] = \left[\begin{array}{c|c} A & r \\ \hline 0 & 1 \end{array}\right] = \left[\begin{array}{ccc|c} a_x & b_x & c_x & r_x \\ a_y & b_y & c_y & r_y \\ a_z & b_z & c_z & r_z \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

[A] is a 3x3 orthonormal matrix of direction cosines, expressing the orientation of the finger (a,b,c) system of Figure 5-7 in terms of the global (x,y,z) system. r is a vector from the origin of the (x,y,z) system to the origin of the (a,b,c) system. If r_f is the vector in Figure 5-7 from f to fp in (a,b,c) coordinates then $[A]\, r_f$ gives the same vector in (x,y,z) coordinates. Consequently, the vector from o to fp in Figure 5-7 is $r = r_b - [A]\, r_f$.

The relationship between two six-element vectors ($d^t = [d_x, d_y, d_z, d_{\theta x}, d_{\theta y}, d_{\theta z}]$) of differential translations and rotations may be expressed as a 6x6 jacobian.

$$d_{bp} = [Jb]\, d_b$$

The jacobian is conveniently written in terms of 3x3 partitions:

$$\underset{(6\times6)}{[Jb]} = \left[\begin{array}{c|c} A^t & A^tR^t \\ \hline \underset{(3\times3)}{0} & \underset{(3\times3)}{A^t} \end{array}\right]$$

[A] is again a 3x3 matrix of direction cosines. In the above example, [A] expresses the orientation of the (l,m,n) coordinate system at bp in Figure 5-7 with respect to the (x,y,z) system. Since [A] is orthonormal it follows that $[A]^t = [A]^{-1}$.

$[R]$ is an antisymmetric cross-product matrix formed from the elements of a vector r, such that if v is a three-component vector (for example, the three rotational components of d_b) then

$$[R]\ v = \begin{bmatrix} 0 & -r_z & r_y \\ r_z & 0 & -r_x \\ -r_y & r_x & 0 \end{bmatrix} \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} = r \times v$$

Since is $[R]$ is antisymmetric, $[R]^t = -[R]$ and $[R]^t v = v^t [R] = v \times r$.

Given the above identities for $[R]$ and $[A]$ the following relationships hold for $[J]$:

$$[Jb]^t = \left[\begin{array}{c|c} A & 0 \\ \hline RA & A \end{array}\right] \qquad [Jb]^{-1} = \left[\begin{array}{c|c} A & RA \\ \hline 0 & A \end{array}\right]$$

$$[Jb]^{-t} = \left[\begin{array}{c|c} A^t & 0 \\ \hline A^t R^t & A^t \end{array}\right]$$

A.2 Matrix Method for Under Determined Finger Motions

For the case in which the motion of the object does not completely determine the motion of the finger, the potential energy may be minimized subject to the nc constraint conditions in $[P]$. The constraint equations, C_i, are formed by multiplying one row of $[P]$ by d_c. The nf auxiliary equations may then be written as [139]

$$\varphi_i = \frac{\partial\, P.E.}{\partial q_i} + \lambda_1 \frac{\partial C_1}{\partial q_i} + \lambda_2 \frac{\partial C_2}{\partial q_i} + \cdots + \lambda_{nc} \frac{\partial C_{nc}}{\partial q_i}$$

These are combined with the constraint equations to provide $nf+nc$ equations for $nf+nc$ unknowns. The equations may be conveniently expressed as

$$\left[\begin{array}{c} d_q \\ \hline l \end{array}\right] = [L]^{-1} \left[\begin{array}{c} g_q \\ \hline d_c \end{array}\right]$$

where l is a column vector of the nc Lagrange multipliers and

$$[L] = \left[\begin{array}{c|c} Kq & P^t \\ \hline P & 0 \end{array}\right]$$

A.3 Differential Jacobians

In Section 5.4.1 the change in the jacobians, $[J]$, as a result of small displacements of the object are considered. These terms, $\Delta[J]$ and $\Delta[J]^t$, result from shifting of the contact area and rolling of the fingers. Products such as $\Delta[J]\cdot d$ contain very small terms and may be ignored, but products such as $\Delta[J]^t\cdot g$ may contain significant terms since the forces, g, may be large. As an example, if the contact area translates and rotates with respect to the object then the change in the jacobian relating g_{bp} and g_b is

$$\Delta[Jb]^t = [Jb']^t - [Jb]^t$$

where $[Jb']^t$ is the jacobian relating to the new position and orientation of the contact area and $[Jb]^t$ is the original jacobian. By writing $[Jb']^t$ and $[Jb]^t$ in terms of partitions, $\Delta[Jb]^t$ is seen to be

$$\left[\begin{array}{c|c} \Delta A & 0 \\ \hline \Delta(RA) & \Delta A \end{array}\right]$$

where $\Delta(RA) = (RA)' - (RA) = [R][A] + [\Delta R][A] + [R][\Delta A] + [\Delta R][\Delta A] - [R][A]$. $[\Delta R][\Delta A]$ contains second order terms, and may be dropped so that $\Delta(RA) \approx [\Delta R][A] + [R][\Delta A]$.

$[\Delta R]$ and $[\Delta A]$ can be written in terms of differential translations and rotations, $(\delta r_x, \delta r_y, \delta r_z, \delta\theta_x, \delta\theta_y, \delta\theta_z)$.

$$[\Delta R] = [R'] - [R] = \begin{bmatrix} 0 & -\delta r_z & \delta r_y \\ \delta r_z & 0 & -\delta r_x \\ -\delta r_y & \delta r_x & 0 \end{bmatrix}$$

$$[\Delta A] = [A'] - [A] = \begin{bmatrix} 0 & -\delta\theta_z & \delta\theta_y \\ \delta\theta_z & 0 & -\delta\theta_x \\ -\delta\theta_y & \delta\theta_x & 0 \end{bmatrix}$$

$[\Delta A]$ and δr are also equivalent to the upper left 3x3 partition and right column respectively of the differential 4x4 homogeneous transform, $[\Delta]$, expressing a small translation and rotation of one coordinate system with respect to another [113].

A.4 Rolling Contact

$\mathbf{r}'_{(s)} = \mathbf{r}_{(s+\delta s)}$ and $\mathbf{u}'_{(s)} = \mathbf{u}_{(s+\delta s)}$ may be expanded in terms of $\mathbf{r}_{(s)}$ as

$$\mathbf{r}'_{(s)} = \mathbf{r}_{(s)} + \delta s\frac{d\mathbf{r}_b}{ds} + \frac{(\delta s)^2}{2!}\frac{d^2\mathbf{r}_b}{ds^2} + \cdots$$

$$\mathbf{u}'_{(s)} = \frac{d\mathbf{r}}{ds} + \delta s\frac{d^2\mathbf{r}}{ds^2} + \cdots$$

Then $\Delta\mathbf{r}$ becomes

$$\Delta\mathbf{r} = \delta s\frac{d\mathbf{r}}{ds} + \frac{(\delta s)^2}{2!}\frac{d^2\mathbf{r}}{ds^2} + \cdots = \delta s\mathbf{u} + \frac{(\delta s)^2}{2!}\frac{d\mathbf{u}}{ds} + \cdots$$

Since the first derivatives of $\mathbf{r}_b$ and $\mathbf{r}_f$ are equal at the initial contact point, subtracting $\Delta\mathbf{r}_b$ - $\Delta\mathbf{r}_f$ gives

$$\Delta\mathbf{r}_b - \Delta\mathbf{r}_f = \frac{(\delta s)^2}{2!}\left(\frac{d^2\mathbf{r}_b}{ds^2} - \frac{d^2\mathbf{r}_f}{ds^2}\right) + \cdots$$

or, $\Delta\mathbf{r}_b - \Delta\mathbf{r}_f \approx 1/2(\delta s)^2$ times the difference in curvature between $\mathbf{r}_{b(s)}$ and $\mathbf{r}_{f(s)}$.

The rotation of the contact point is given by the vector $(\mathbf{u}_b \times \mathbf{u}_b')$ and the rotation of the fingertip is given by $(\mathbf{u}_f' \times \mathbf{u}_b')$. Expanding $\mathbf{u}_f'$ and $\mathbf{u}_b'$ in terms of $\mathbf{r}_{(s)}$ and discarding third and higher derivatives of $\mathbf{r}$ gives

$$\mathbf{u}_b \times \mathbf{u}_b' = \mathbf{u}_b \times \left(\mathbf{u}_b + \frac{d\mathbf{u}_b}{ds}\delta s\right) = (0) + \delta s\left(\frac{d\mathbf{r}_b}{ds} \times \frac{d^2\mathbf{r}_b}{ds^2}\right)$$

and

$$\mathbf{u}_f' \times \mathbf{u}_b' = (\mathbf{u}_f \times \mathbf{u}_b) + \delta s\left(\mathbf{u}_f \times \frac{d\mathbf{u}_b}{ds}\right) + \delta s\left(\frac{d\mathbf{u}_f}{ds} \times \mathbf{u}_b\right) + (\delta s)^2\left(\frac{d\mathbf{u}_f}{ds} \times \frac{d\mathbf{u}_b}{ds}\right)$$

$$= (0) + \delta s\left(\left(\frac{d^2\mathbf{r}_f}{ds^2} - \frac{d^2\mathbf{r}_b}{ds^2}\right) \times \mathbf{u}\right) + (\delta s)^2\left(\frac{d^2\mathbf{r}_f}{ds^2} \times \frac{d^2\mathbf{r}_b}{ds^2}\right)$$

where $\mathbf{u} = \mathbf{u}_b = \mathbf{u}_f$ at the initial contact point.

For the example in Section 5.6.3, where the object surface is flat and the fingertip is a segment of a circular arc, as in Figures 5-21 and 5-22, the rolling equations become

$$\mathbf{r}_f = (r_f \sin\theta_f)\mathbf{i} - (r_f \cos\theta_f)\mathbf{j}, \quad \mathbf{r}_b = (\tfrac{w}{2}\tan\theta_b)\mathbf{i} + \tfrac{w}{2}\mathbf{j}.$$

where θ_f is related to s as

$$\frac{d\theta_f}{ds} = 1/r_f$$

For $\theta_f = \theta_b = 0$ at the initial contact point, equations (5.14)-(5.17) become

$$\Delta \mathbf{r}_b = (r_f \delta\theta_f)\,\mathbf{i} + 0\,\mathbf{j}$$

$$\Delta \mathbf{r}_b - \Delta \mathbf{r}_f = 0\,\mathbf{i} - \frac{r_f(\delta\theta_f)^2}{2}\,\mathbf{j} \approx 0$$

$$\mathbf{u}_b \times \mathbf{u}_b' = 0$$

$$\mathbf{u}_f' \times \mathbf{u}_b' = -\delta\theta_f \mathbf{k}$$

A.5 Details for Examples in Section 5.6

Summary of matrix equations for left finger — point-contact example

1. $d_{bp} = [M][Jb]d_b$
2. $d_{fp} = [Jfq]d_q$
3. .

$$\begin{bmatrix} d_q \\ \hline \lambda_1 \\ \lambda_2 \end{bmatrix} = [L]^{-1} \begin{bmatrix} g_q \\ \hline dm \\ dn \end{bmatrix}$$

4. $\delta g_q = [Kq]d_q$
5. $[Cfp] = [Jfq][Kq]^{-1}[Jfq]^t$
6. $[Cfc]$ = non-singular portion of $[Cfp]$
7. d_{fc} = subset of d_{fp} corresponding to $[Cfc]$
8. $\delta g_{fc} = [Cfc]^{-1}d_{fc}$
9. $\delta g_{fp} = \delta g_{fc} + \Delta[Jf]^{-t}g_f$
10. $\delta g_b = [Jb]^t[M]^t\,\delta g_{fp}$