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Running With Symmetry

Abstract

Symmetry can simplify the control of dynamic legged systems. In this paper, the symmetries studied describe motion of the body and legs in terms of even and odd functions of time. A single set of equations describes symmetric running for systems with any number of legs and for a wide range of gaits. Techniques based on symmetry have been used in laboratory experiments to control machines that run on one, two, and four legs. In addition to simplifying the control of legged machines, symmetry may help us to understand legged locomotion in animals. Data from a cat trotting and galloping on a treadmill and from a human running on a track conform reasonably well to the predicted symmetries.

1. Introduction

Running is a series of bouncing and ballistic motions that exert forces on the body during every stride. The bouncing motions are caused by the vertical rebound of the body when the legs push on the ground, and the ballistic motions occur between bounces when the body is airborne. If a legged system is to keep its forward running speed fixed and its body in a stable upright posture despite these motions, then the net acceleration of the body must be zero over each entire stride. This requires that the torques and horizontal forces exerted on the body by the legs must integrate to zero over each stride and that the vertical forces must integrate to the body's weight times the duration of the stride. This is equally true for running machines and for running animals.

Although there are many patterns of body and leg motion that can satisfy these requirements, a particularly simple solution arises when each variable has an even or odd symmetry during the time a foot is in contact with the ground.

$$\text{Body Symmetry} \quad \begin{cases} x(t) = -x(-t), \\ z(t) = z(-t), \\ \phi(t) = -\phi(-t), \end{cases} \tag{1}$$

$$\text{Leg Symmetry} \quad \begin{cases} \theta(t) = -\theta(-t), \\ r(t) = r(-t). \end{cases} \tag{2}$$

x, z, and ϕ are the forward position, vertical position, and pitch angle of the body, and θ and r are the angle and length of the leg, all measured in the *sagittal plane** (see Fig. 1). For simplicity, t and x are defined so that $t = 0$ halfway through the stance phase, and $x(0) = 0$. These symmetry equations specify that forward body position, body pitch angle, and leg angle are each odd functions of time throughout the stance phase, and that body elevation and axial leg length are even functions of time. The symmetry also requires that the actuators operate with even and odd symmetry:

$$\text{Actuator Symmetry} \quad \begin{cases} f(t) = f(-t), \\ \tau(t) = -\tau(-t), \end{cases} \tag{3}$$

where τ is the torque exerted about the hip and f is the force exerted along the leg axis.

These symmetries are significant because they result in accelerations of the body that are odd functions of time throughout a stride. Odd functions integrate to zero over symmetric limits, leaving the forward running speed, body elevation, and body pitch angle unchanged from one stride to the next.

We first recognized the value of symmetry when exploring control for machines that balance as they hop on one leg (Raibert and Brown 1984; Raibert,

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* The sagittal plane for animals is defined by the fore–aft and up–down directions.

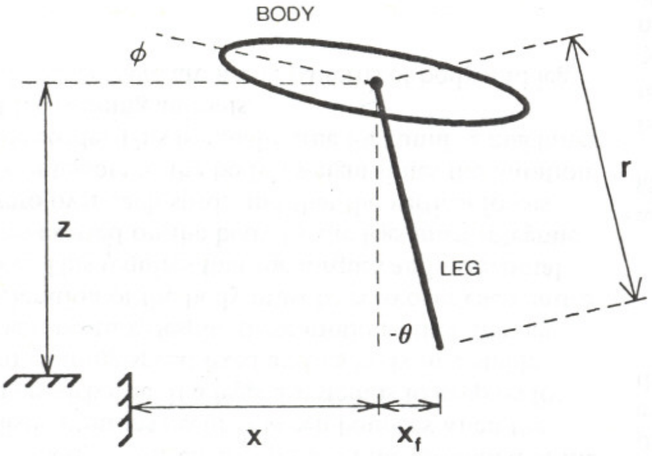


Fig. 1. Definition of variables used in symmetry equations. Positive τ acts about the hip to accelerate the body in the positive ϕ direction. Positive f acts along axis of the leg and pushes the body away from the ground.

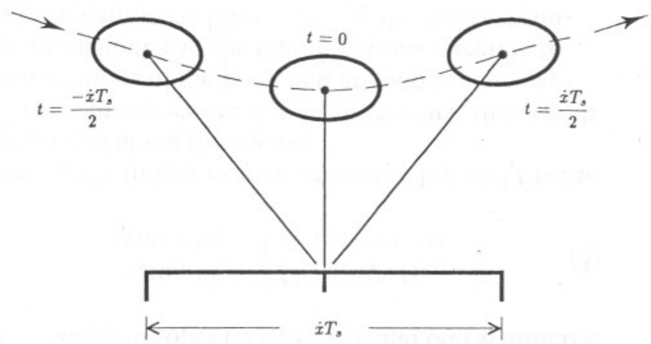


Fig. 2. When the foot is placed on the neutral point, there is a symmetric motion of the body. The figure depicts running from left to right. The left-most drawing shows the configuration just before the foot touches the ground, the center drawing shows the configuration halfway through stance when the leg is maximally compressed and vertical, and the right-most drawing shows the configuration just after the foot loses contact with the ground.

Brown, and Chepponis 1984). Like an inverted pendulum, a one-legged system tips and accelerates when its point of support is not located directly below the center of mass. Symmetric motion ensures that tipping and acceleration in one direction is balanced by equal tipping and acceleration in the opposite direction (see Fig. 2). The control system we implemented for the one-legged machines uses a knowledge of the dynamics to manipulate the machine's initial configuration on each step to produce a symmetric motion during the ensuing support phase. Symmetry simplifies the control because it frees the control system from regulating the details of the trajectory—the details are determined passively by the mechanical system. One-legged hopping machines that run and balance using these techniques are described in Raibert and Brown (1984) and Raibert, Brown, and Chepponis (1984). The approach was recently extended to the control of a running biped (Hodgins, Koechling, and Raibert, 1986) and to a trotting quadruped (Raibert, Chepponis, and Brown 1986).

This paper introduces motion symmetry in the context of one-legged systems and then generalizes to more complicated cases. Symmetry is particularly simple for one-legged machines because only one leg provides support at a time, each support interval is isolated in time by periods of ballistic flight, and the hip is located at the center of mass. After reviewing the one-legged case, we consider motions that span several support intervals and the use of several legs for support during a single support interval. One set of symmetry equations applies for one, two, and four legs, and for gaits that use legs singly and in combination.

In addition to suggesting simple control for legged robots, the symmetries developed in this paper may help us to understand the control mechanisms at work in running animals. Hildebrand (1965, 1966, 1968, 1976) established the importance of symmetry in animal locomotion when he observed that the left half of a horse often uses the same pattern of footfalls as the right half, but is 180 degrees out of phase. He devised a simple and elegant characterization of the symmetric walking and running gaits for a variety of quadrupeds, using just two parameters: the phase angle between the front and rear legs and the duty cycle of the legs. By mapping each observation of symmetric behavior into a point in phase/duty-cycle space, Hildebrand was able to classify systematically gaits for over 150 quadruped genera.

Rather than look at relationships between the footfalls of the left and right legs as Hildebrand did, I measured the trajectories of the feet with respect to the body and the trajectory of the body through space in the sagittal plane. Data for the trotting and galloping cat and for the running human show that they sometimes move as the symmetries predict.

2. Mechanics of Symmetry

A number of simplifications ease the analysis of symmetry. The analysis is based on a model that is restricted to move in the plane, with massless legs and no losses anywhere in the system. The body is a rigid object that moves fore and aft and up and down and

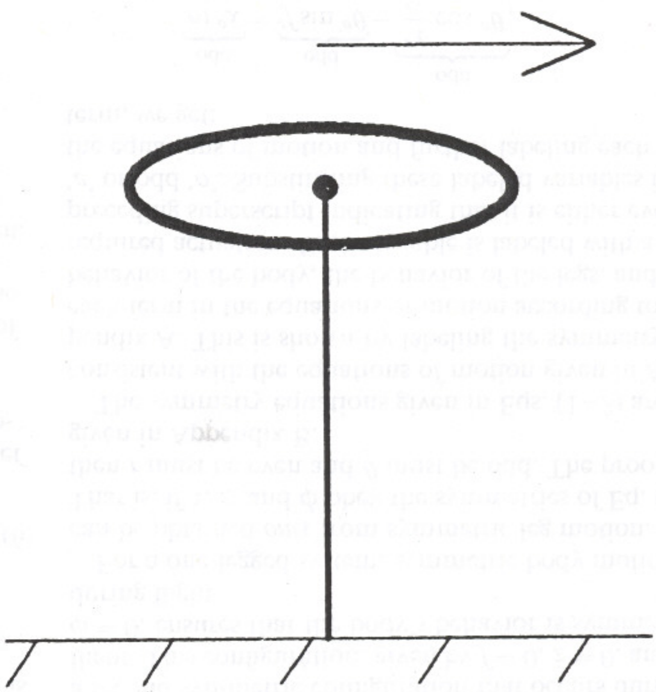


Fig. 3. Symmetric configuration of a one-legged system halfway through stance, when it has fore-aft symmetry (left-right as shown in diagram) as well as symmetry moving forward and backward in time. The vertical velocity is zero, the support point is located directly under the center of mass, and the body is upright: $\dot{\theta}(0) = x_f(0) = \phi(0) = 0$.

that pitches in the plane, with position and orientation given by $[x\ z\ \phi]$. Each leg is a single member that pivots about its hip on a hinge-type joint and that lengthens and shortens by telescoping. The length of a leg and its angle with respect to the vertical are given by $[r\ \theta]$. A foot at the end of each leg provides a single point of support. Friction between a foot and the ground prevents the foot from sliding when there is contact. A foot in contact with the ground acts mechanically like a hinge joint.

Each leg actuator exerts a force f along the leg's axis between the body and the ground. Positive f accelerates the body away from the ground, and, because the feet are not sticky, $f \geq 0$. This force is zero when there is no contact between the foot and the ground. Normally the leg is springy in the axial direction, in which case f is a function of leg length. A second actuator acts at the hip, generating a torque τ between the leg and the body. Positive τ accelerates the body in the positive ϕ direction. Equations of motion for this sort of model are given in Appendix A.

In normal operation, the models follow a regular pattern of activity, alternating between periods of support and periods of flight. The transition from flight to support is called *touchdown* and the transition from support to flight is called *lift-off*. During support, a foot remains stationary and the leg exerts a combination of vertical and horizontal forces on the body. Because legs are springy, the body's vertical motion is an elastic rebound that returns the system to the flight phase after each collision with the ground. Once airborne, the body follows a ballistic trajectory. The body may derive support from one or more legs during a single support interval depending on the number of legs in the system and the gait. Because the legs have no mass and the entire system is lossless, bouncing motions continue undiminished without an external source of energy.

2.1. Symmetric Motion with One Leg

Imagine that at time $t = 0$ the foot of a one-legged system is located directly below the center of mass, the body is upright, and the velocity of the body is purely horizontal: $\theta = 0$, $\phi = 0$, and $\dot{z} = 0$. Figure 3 shows this configuration. Because it has left-right symmetry and there are no losses, the system's expected behavior proceeding forward in time is precisely the same as its past behavior receding backward in time, but with a reflection about the line $x = 0$. This behavior is described by the body symmetry equations, which state that $x(t)$ and $\phi(t)$ are odd functions of time and $z(t)$ is an even function. Because the body moves along a symmetric trajectory with respect to the origin and because the foot is located at the origin during support, the body symmetry equations imply that the foot's motion is symmetric with respect to the body, which gives the leg symmetry equations, Eq. (2).

Symmetric motion of the body and legs requires symmetric actuation, as given in Eq. (3). From the equations of motion (see Appendix A), we see that hip torque is the only influence on body pitch angle, so odd ϕ implies odd τ. With the evenness and oddness of the other variables specified, f must be even to satisfy the equations of motion.

A locomotion system operates in steady state when the state variables, measured at the same time during each stride cycle, do not vary from stride to stride. The state variables of interest are the body's forward velocity, vertical position, vertical velocity, pitch angle, and pitch rate. With the state vector **S** representing these variables, $\mathbf{S} = [\dot{x}\ z\ \dot{z}\ \phi\ \dot{\phi}]$, steady state is defined by

$$\mathbf{S}(t) = \mathbf{S}(t + T), \tag{4}$$

where T is the duration of one stride.

Symmetric body and leg motion results in *steady-state locomotion.** For the forward speed to remain unchanged from stride to stride, the horizontal force f_x acting on the body must integrate to zero over a stride:

$$\int_{stride} f_x\, dt = 0. \tag{5}$$

Assume that $f_x = 0$ during flight and that the forward speed does not change. From the equations of motion we get $f_x = f \sin\theta - \left(\frac{\tau}{r}\right)\cos\theta$ during stance, which is an odd function because f and r are even and τ and θ are odd. Therefore

$$\dot{x}(t_{lo}) - \dot{x}(t_{td}) = \int_{t_{td}}^{t_{lo}} f_x\, dt = 0. \tag{6}$$

This confirms that symmetric motion provides no net horizontal force on the body, and running speed proceeds in steady state from stride to stride.

The vertical position and velocity also proceed in steady state for a symmetric motion. The elevation of the body is an even function of time during stance, so $z(t_{lo,i}) = z(t_{td,i})$ and $\dot{z}(t_{lo,i}) = -\dot{z}(t_{td,i})$. During flight, the body travels a parabolic trajectory that is also even, if we specify $t = 0$ halfway through flight: $z(t_{td,i+1}) = z(t_{lo,i})$ and $\dot{z}(t_{td,i+1}) = -\dot{z}(t_{lo,i})$. Consequently, $z(t_{td,i}) = z(t_{td,i+1})$ or $z(t_{td}) = z(t_{td+T})$, which is the steady-state condition on z, and $\dot{z}(t_{td}) = \dot{z}(t_{td+T})$, which is the steady-state condition on $\dot{z}$.

The torque acting on the body is zero during flight and an odd function during stance, so the body pitch rate undergoes zero net acceleration during stance, $\dot{\phi}(t_{lo}) = \dot{\phi}(t_{td})$. This satisfies the steady-state condition on $\dot{\phi}$. For the pitch angle of the body to proceed in steady state, its value at the end of flight must be equal and opposite to its value at the beginning of the flight phase. Assuming that symmetry holds during stance so that $\phi_{lo} = -\phi_{td}$ and that no torques act on the body during flight, a repeating pattern requires that

$$\frac{\dot{z}(t)}{-g} = \frac{\phi(t)}{\dot{\phi}(t)}, \tag{7}$$

where g is the acceleration of gravity. This constraint prescribes the relationship among pitch angle, pitch rate, and vertical velocity needed for steady-state running. It is trivially satisfied when there is no pitching motion, $\phi(t) = 0$ and $\dot{\phi}(t) = 0$. Equation (7) results in a second symmetric configuration that occurs during flight. This configuration, given by $f = 0$, $\dot{z} = 0$, and $\phi = 0$, ensures that the body's behavior is symmetric during flight.

For a one-legged system, symmetric body motion can be obtained *only* from symmetric leg motion. That is, if x, z, and ϕ obey the symmetries of Eq. (1), then r must be even and θ must be odd. The proof is given in Appendix B.

The symmetry equations given in Eqs. (1–3) are consistent with the equations of motion given in Appendix A. This is shown by labeling the symmetry for each term in the equations of motion according to the behavior of the body, the behavior of the legs, and the required actuation. Each variable is labeled with a preceding superscript indicating that it is either even 'e' or odd 'o'. Substituting these labeled variables in the equations of motion and further labeling each term, we get:

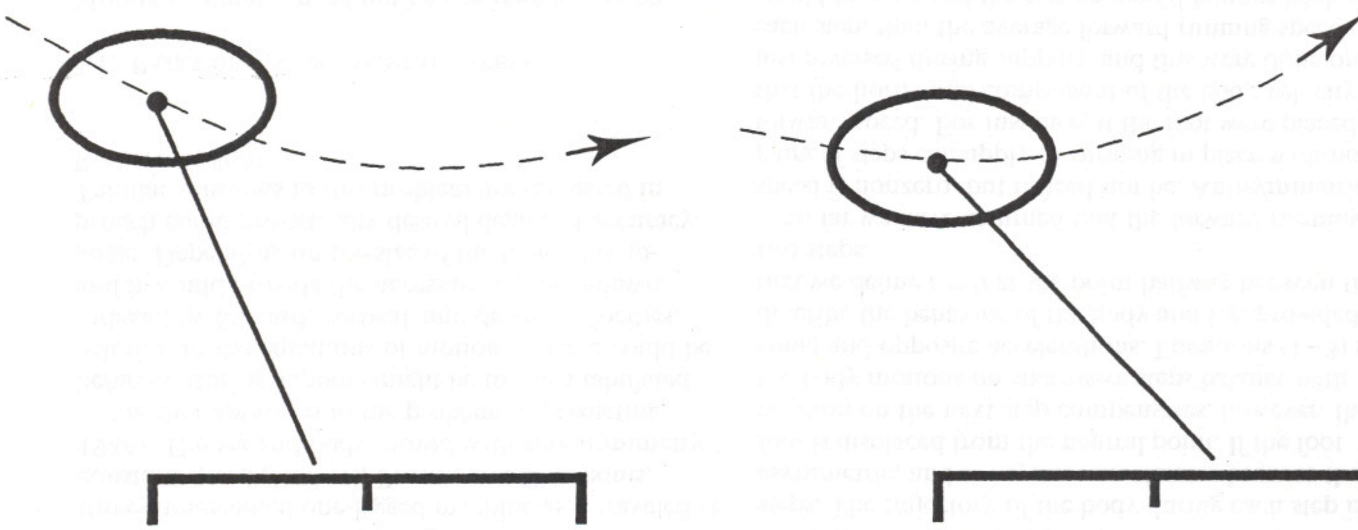
$$\overbrace{m\,{}^{o}\ddot{x}}^{\text{odd}} = \overbrace{{}^{e}\!f \sin {}^{o}\theta}^{\text{odd}} - \overbrace{\frac{{}^{o}\tau}{{}^{e}r}\cos {}^{o}\theta}^{\text{odd}}, \tag{8}$$

$$\overbrace{m\,{}^{e}\ddot{y}}^{\text{even}} = \overbrace{{}^{e}\!f \cos {}^{o}\theta}^{\text{even}} + \overbrace{\frac{{}^{o}\tau}{{}^{e}r}\sin {}^{o}\theta}^{\text{even}} - \overbrace{mg}^{\text{even}}, \tag{9}$$

$$\overbrace{I\,{}^{o}\ddot{\phi}}^{\text{odd}} = \overbrace{{}^{o}\tau}^{\text{odd}}. \tag{10}$$

* A trajectory that provides steady-state locomotion is one that provides a nominal motion that would repeat from cycle to cycle if there were no disturbances. It does not mean that there are restoring forces that will return the system to the trajectory if it deviates as the result of a disturbance. Restoring forces are also required for stability once the nominal trajectory has been determined. Asymmetry in the motion is a source of such restoring forces.

Fig. 4. Asymmetric trajectories. Displacement of the foot from the neutral point accelerates the body by skewing its trajectory. When the foot is placed behind the neutral point, the body accelerates forward during stance (left). When the foot is placed forward of the neutral point, the body accelerates backward during stance (right). Dashed lines indicate the path of the body and solid horizontal lines under each figure indicate the CG-print.



A similar procedure shows that the symmetries are consistent with the equations of motion for multi-legged systems as well.

3. Generating Symmetric Motions

The discussion has focused on the nature and value of symmetric motion without addressing the generation of symmetric motion. What action must a control system take to produce symmetric behavior? Recall that a legged system moves with symmetry if θ, $\dot{z}$, and ϕ all equal zero at the same time during support, but the control system must commit the foot to a position on the ground before touchdown when neither $\dot{z}$ nor ϕ is zero. The task of orchestrating such a rendezvous is to predict where the center of mass will be when the body's vertical velocity and pitch angle are both zero.

General solutions to this problem are not known. The difficulty in accomplishing this task is that placement of the foot influences the path of the body. If we had an expression for the body's trajectory during support as a function of $\dot{x}(t_{td})$, $\dot{z}(t_{td})$, and $\theta(t_{td})$—a solution to the equations of motion—then we could solve for the desired foot placement. We have not found a closed-form expression for the path of the body during support, even for simple models.

Despite the lack of a general solution, approximate solutions exist for gaits that use just one leg for support at a time, the one-foot gaits. The simplest approximate solution assumes that forward speed is constant during support and that the period of support T_s is constant, depending on only the spring mass characteristics of the leg and body. These approximations estimate the length of the *CG-print,* the forward distance traveled by the center of mass during support, as $\dot{x}T_s$:

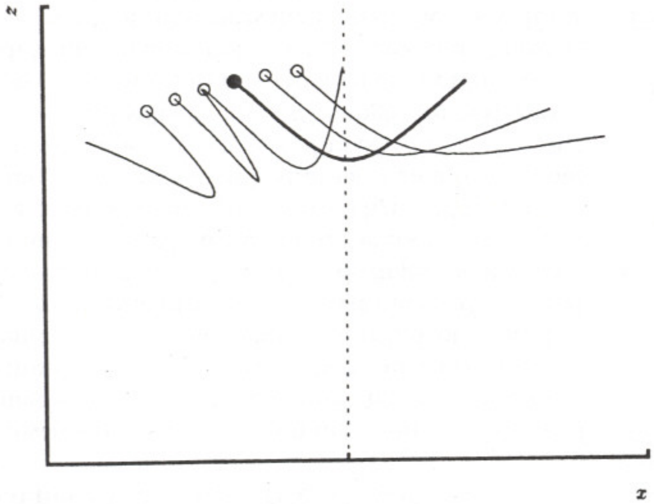
$$x_f = \frac{\dot{x}T_s}{2} - k_{\dot{x}}(\dot{x} - \dot{x}_d), \tag{11}$$

where

x_f	is the forward displacement of the foot with respect to the projection of the center of mass,
$\dot{x}$	is the forward velocity of the body,
$\dot{x}_d$	is the desired forward velocities of the body,
T_s	is the duration of a support period, and
$k_{\dot{x}}$	is a gain.

The first term in Eq. (11) is the neutral foot position that provides symmetry. The second term introduces asymmetry that accelerates the system to correct for errors in running speed, as shown in Fig. 4. It displaces the foot from the neutral point to skew the pattern of body motion. A set of systematically skewed motions is shown in Fig. 5. These displacements accelerate the body to stabilize its motion against disturbances and to change running speed. Control systems for one-,

Fig. 5. Path of the body during stance for several forward foot positions. Only the neutral foot position results in a symmetric body trajectory (bold), whereas those to either side are skewed, either forward or backward. The initial forward speed is the same for each trajectory. The circles indicate the location of the body at touchdown, and the origin is the foot position. These data are from simulations of a model with a linear leg spring. Adapted from Stentz (1983).



two-, and four-legged machines use this approximation to choose a forward displacement for the foot during flight (Raibert 1986*b*).

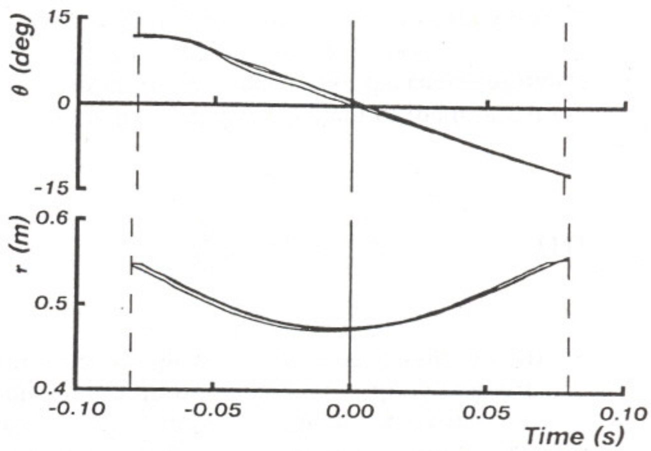
Our experience with this approximation is that it provides good symmetry at low and moderate running speeds. The data shown in Fig. 6 were recorded from a three-dimensional one-legged machine as it traveled at constant speed (Raibert, Brown, and Chepponis, 1984). The leg and body moved with good symmetry.

Another approach to the problem of predicting behavior during support might be to use a tabulated solution to the equations of motion. A table could be indexed by forward, vertical, and desired velocities, and it would provide the necessary leg touchdown angle. Depending on the size of the table, this approach could provide any desired degree of accuracy. Tabular solutions to this problem are discussed in Raibert (1986*b*).

3.1. Pairs of Antisymmetric Steps

Motion symmetry need not be confined to just one step. Although we have concentrated on symmetry that applies on a step-by-step basis, the symmetries apply equally well when pairs of steps produce complementary accelerations, with the symmetry distributed over more than one support interval. This case is discussed next.

Fig. 6. Symmetry data recorded from a physical, 3-D, one-legged, hopping machine. The behavior of the machine obeys the symmetry equations when the foot is placed on the neutral point. Data for three consecutive support intervals are superimposed. The leg is longer at lift-off than at touchdown because it lengthens during support to provide thrust that compensates for various mechanical losses in the system. The time axes were adjusted so that $t = 0$ halfway through the support interval, and the x-origin was adjusted so that $x(t = 0) = 0$. Running speed is about 1.6 m/s. Dashed vertical lines indicate touchdown and lift-off.

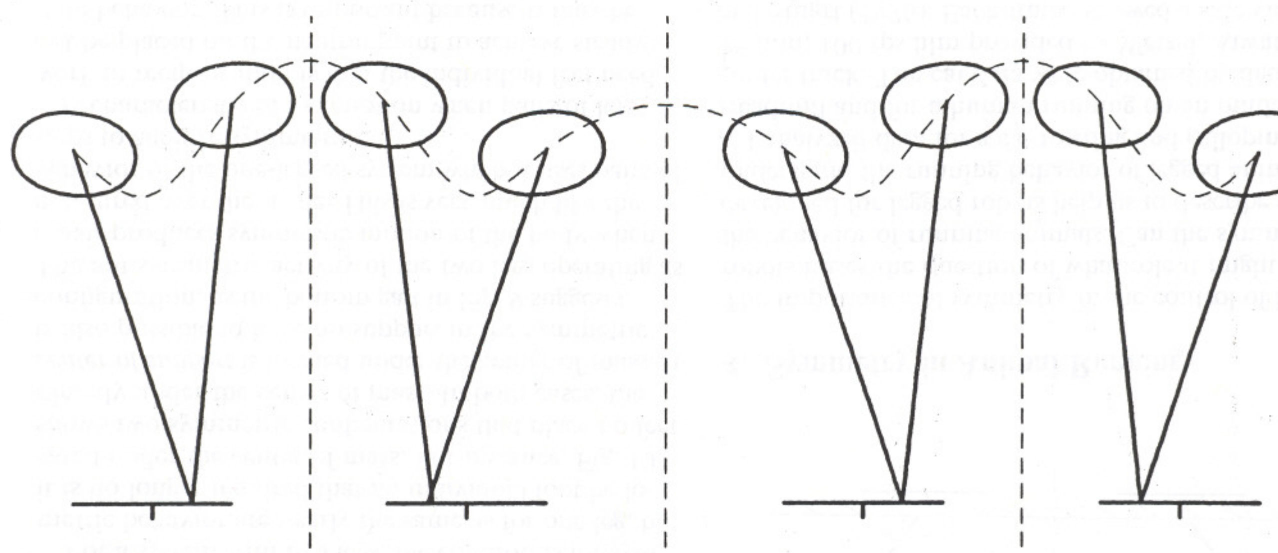


Suppose that single support periods deviate from symmetry but that two sequential support periods each deviate from symmetry in a complementary fashion. Figure 7 shows a sequence of such antisymmetric steps. The trajectory of the body during each step is asymmetric, and the system accelerates because the foot is displaced from the neutral point. If the foot position on the next step compensates, however, then the body motions on successive steps balance with equal and opposite accelerations. Equations (1 – 3) still describe the behavior of the body and leg, provided that we define $t = 0$ at the point halfway between the two steps.

So far we have assumed that the forward running speed is nonzero, but it need not be. Antisymmetric pairs of steps can apply to running in place with no forward speed. For instance, if the foot were placed so that the horizontal component of the body velocity is just reversed during support, and this were done on each step, then the average forward running speed would be zero and the system would bounce back and forth on each step. This is just the sort of behavior observed in the frontal plane of the human and the pacing quadruped.

Figure 8 presents data from a physical demonstration of symmetry distributed over a pair of steps for which the forward running speed is zero. To generate

Fig. 7. Pairs of antisymmetric steps. If the foot is positioned behind the neutral point on one step, and in front of it on the next step, then the pair of steps may have symmetry that stabilizes the forward running speed, even though the motion during each step is no longer symmetric. We redefine the stride to include the symmetric pair of steps. The body and leg are drawn once for each touchdown and lift-off. The vertical dashed lines indicate the planes of symmetry, which occur halfway through the strides and between the strides.



these data, we modified the control algorithm for a physical one-legged hopping machine to add an offset Δx to the desired foot placement on every even-numbered hop and to subtract Δx on every odd-numbered hop. For small values of Δx, the system hopped from side to side with no net forward acceleration. The system maintained its balance, provided that the offset of the foot was small enough so that the system did not tip over entirely before the next step.

3.2. Symmetry with Several Legs

A system with two legs can run with a variety of gaits. The two legs can operate precisely in phase, precisely out of phase, or with intermediate phase. Figure 9 shows several examples that differ with regard to the amount of body pitching, the variation in forward running speed within a stride, and the degree of temporal overlap in the support provided by the two legs. In each case, however, symmetric body and leg motion results in steady-state locomotion.

The body symmetries for a system with several legs are the same as for one leg, but the leg and actuator symmetries are modified slightly. Each leg and actuator variable, θ, r, τ, and f, have the same meanings as before but with subscripts to distinguish among the individual legs:

$$\theta_j(t) = -\theta_k(-t), \tag{12}$$

$$r_j(t) = r_k(-t), \tag{13}$$

$$\tau_j(t) = -\tau_k(-t), \tag{14}$$

$$f_j(t) = f_k(-t). \tag{15}$$

For a system with two legs, $j = 1$ and $k = 2$. For four legs, two pairings are possible: $j = [1, 4]$ and $k = [2, 3]$ or $j = [1, 4]$ and $k = [3, 2]$, depending on the gait, where 1 is left front, 2 is left rear, 3 is right rear, and 4 is right front.

Symmetric body motion no longer requires an individual leg to move with a symmetry of its own. Instead, the behavior of one leg is linked to the behavior of another leg, so that they operate with reciprocating symmetry. This frees the variables describing any one leg to take on arbitrary functions of time while preserving the symmetric forces and moments impinging on the body during support. These motion symmetries apply when legs operate in unison, when legs have different but overlapping support periods, and when the legs provide support separately. As before, the equations that describe leg motion apply only when $f > 0$, so it does not matter how the legs move when

Fig. 8. Symmetric pairs of steps. The curves show the recorded path of the body for a physical, one-legged hopping machine hopping in place. The control algorithms were those described by Raibert, Brown, and Chepponis (1984), but an offset, Δx, was added to the foot position on even-numbered hops and subtracted on odd-numbered hops. The magnitude of Δx was set to two different values, shown separately in the two curves. The plot shows the motion of the body in a vertical plane that contains the foot offset.

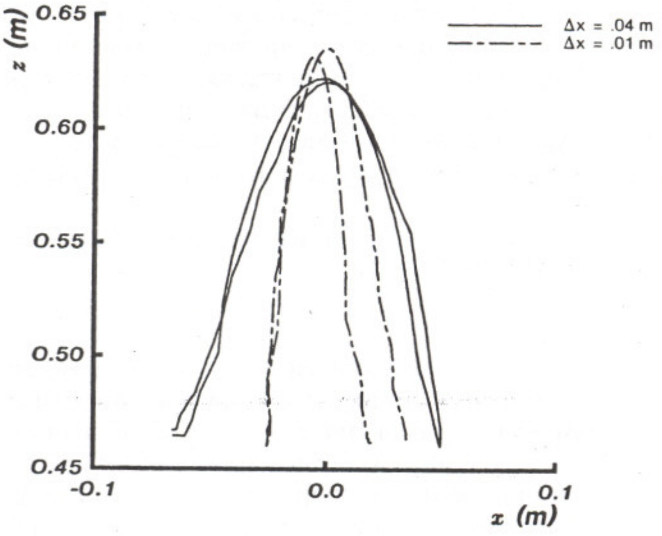
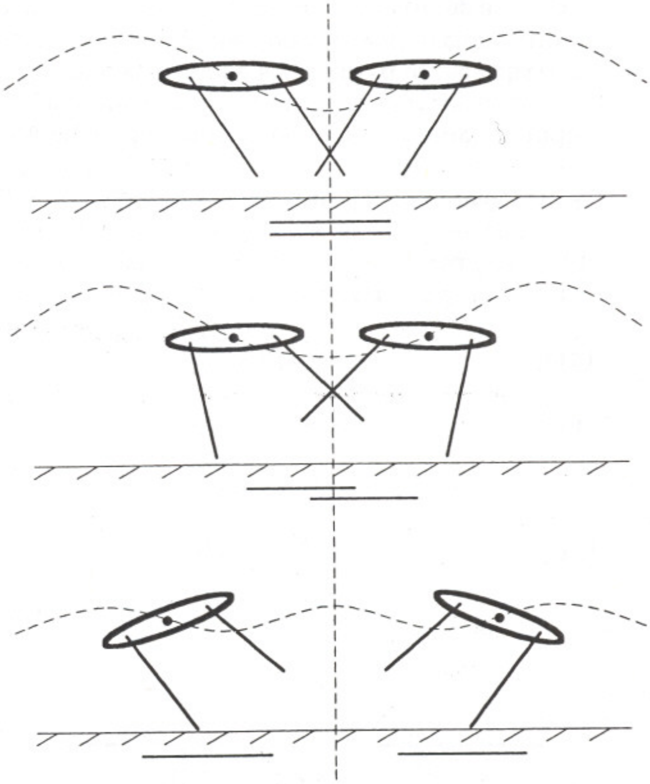


Fig. 9. Running with two legs separated by a long body. Symmetry can be achieved when both feet provide support simultaneously, when there is partial overlap in the support periods, and when the legs provide support in sequence. These three cases are distinguished by the phasing of the legs. It may be difficult to place the feet on the neutral points when the hips are widely separated. Displacements of the feet from the neutral points influence pitching of the body and the duration of each flight phase.



they are not touching the ground. Equations (12–15) reduce to the one-legged case when $j = k = 1$.

For a system with two legs, the conditions for symmetric behavior are nearly the same as for one leg, but it is no longer required that an individual foot be located under the center of mass. For instance, Fig. 10 shows two symmetric configurations that place no feet directly under the center of mass. In both cases, the *center of support* is located under the center of mass. It is also possible to have no support in the symmetric configuration, as the bottom gait in Fig. 9 suggests. The antisymmetric activity of the two legs operating as a pair produces symmetric motion of the body when measured over the stride. This is very much like the behavior of the one-legged system when it uses pairs of steps to achieve symmetry.

A characteristic of locomotion when pairs of legs work in reciprocation is that the individual feet need not be placed on the neutral point to achieve steady-state behavior. This is important because it may be difficult for a legged system to reach far enough under the center of mass when the hips and shoulders are located at the extremes of a long body. This situation arises in the sagittal plane for the quadruped bound and gallop and to a lesser extent in the frontal plane for the quadruped pace.

4. Symmetry in Animal Running

The importance of symmetry in the control of legged robots raises the question of what role it might play in the behavior of running animals. Can the symmetries developed for legged robots help us to describe and understand the running behavior of legged animals?

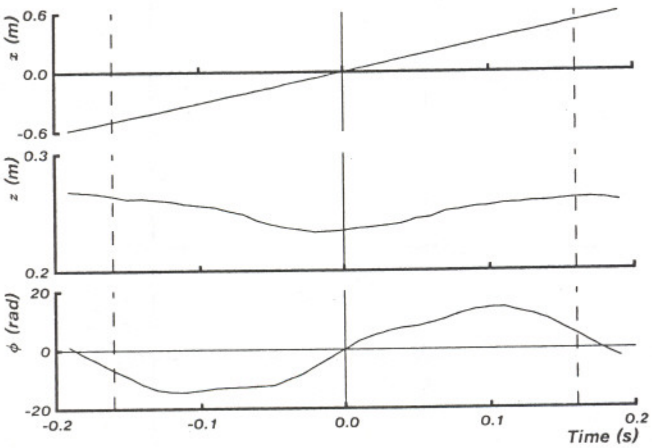
I analyzed data for a cat trotting and galloping on a treadmill and for a human running on an outdoor cinder track. The cat data were obtained by digitizing 16 mm, 100 fps film provided by Wetzel, Atwater, and Stuart (1976). Each frame showed a side view of the cat on the treadmill and a 1-ms counter used to calibrate the film speed. Treadmill markers spaced at 0.25-m intervals provided a scale of reference and permitted registration of each frame. Small circular markers attached to the cat's skin made the digitizing easier. Running speeds with respect to the treadmill

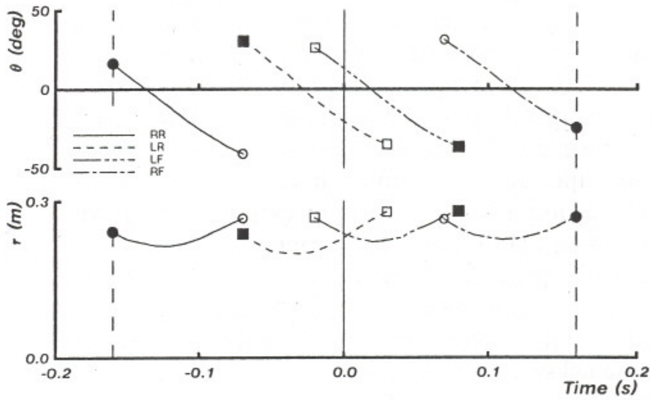
Fig. 10. Symmetric configuration during support for two legs. Configuration of two-legged systems halfway through the support interval. The center of support *is located under the center of mass, vertical velocity is zero, and the body is upright:* $\theta_i + \theta_j = \dot{z} = \phi = 0$.

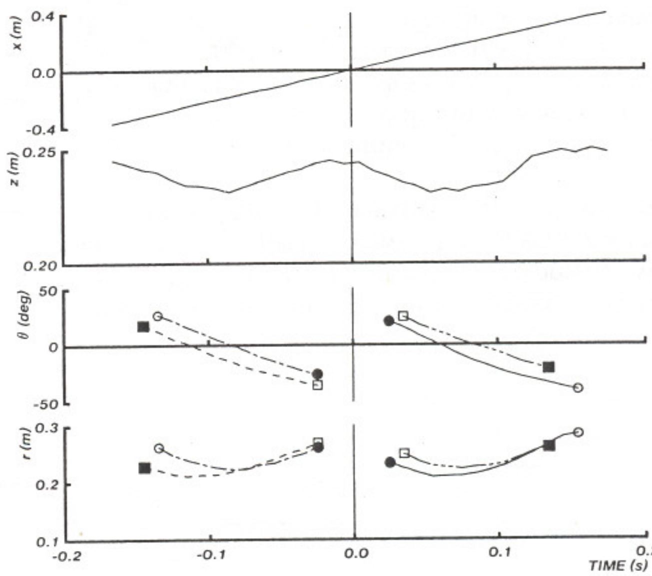
Fig. 11. Body motion of the galloping cat. Data are shown for one stride of a cat running on a treadmill with a rotary gallop. According to symmetry theory, forward body position x *and body pitch angle* ϕ *should each have odd symmetry, and body height* z *should have even symmetry. The symmetry displayed in these plots is good. Dashed vertical lines indicate the beginning and end of the stance phase. Solid vertical line indicates the symmetry point, when* $t = 0$.

Fig. 12. Leg motion for the galloping cat. Leg angle θ *should have odd symmetry and leg length* r *should have even symmetry. Symmetry in behavior of the legs is found when they are considered in reciprocating pairs, e.g.,* $\theta_{RR}(t) = -\theta_{RF}(-t)$ *and* $r_{RR}(t) = r_{RF}(-t)$. *Symbols indicate pairs of points that should have symmetric positions with respect to the origin (for odd symmetry), or the z-axis (for even symmetry). Both leg angle and leg length show very good symmetry. Data for each leg are shown only when its foot touches the support surface. Dashed vertical lines indicate the beginning and end of the stance phase. Solid vertical line indicates the symmetry point, when* $t = 0$. *The data are from the same stride as in Fig. 11.*

Fig. 13. Data for the cat trotting on a treadmill. The left front and hind legs form one pair of legs that operate in reciprocating symmetry, and the right front and rear legs form the other reciprocating pair. Running speed was 2.2 m/s.





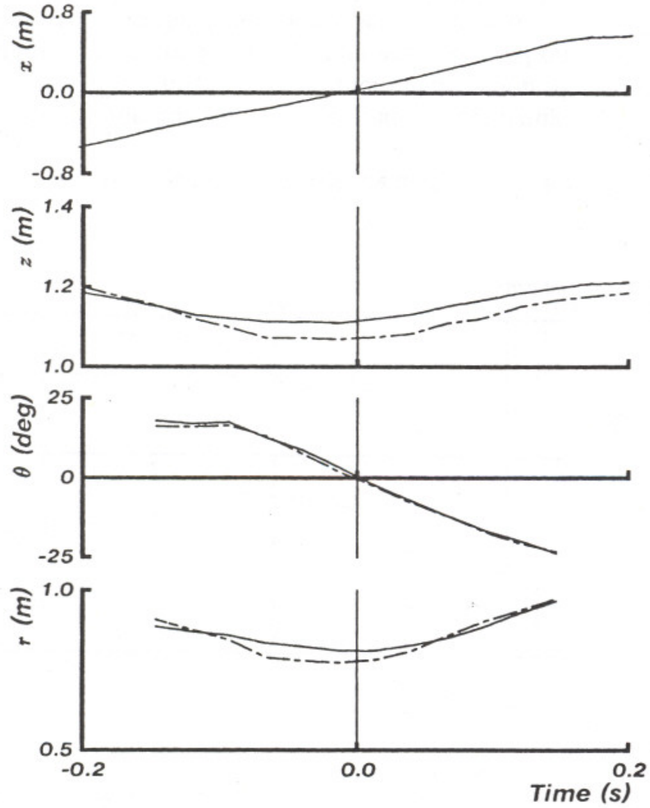


surface were about 2.2 m/s for trotting and 3.1 m/s for galloping.

The human measurements were made by digitizing 16 mm film of a runner on the semicircular section of an outdoor cinder track. The camera was mounted on a tripod located at the center of the semicircle and panned to track the runner. Ground markers spaced at 1.0-m intervals provided scale and registration as before. Running speed was about 3.8 m/s.

In digitizing both the cat and the human data, the point of support provided by each foot was estimated visually. A straight line from this point to the hip, or shoulder for the cat's front legs, was used to find the leg length r and the leg angle θ. The center of mass of the cat was taken as the midpoint between the shoulder and the hip. The pitch angle of the body was the angle between the horizontal and the line connecting shoulder to hip, offset so that $\phi(0) = 0$. These mea-

Fig. 14. Data for one stride of a human running on an outdoor cinder track. Data for the right (stippled) and left legs are superimposed. Running speed was 3.8 m/s. Subject MHR.

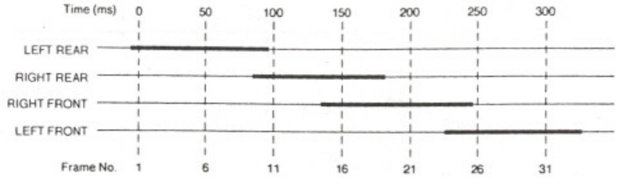


surements provided three parameters of the body's motion: its forward position, vertical position, and pitch angle $[x\ z\ \phi]$. The measurements also gave two parameters of each leg's motion: its length and angle with respect to the vertical $[r\ \theta]$. In addition to information about the timing of footfalls, these measurements provided information about where on the ground the feet were placed with respect to the body and how the body itself moved.

Data for one stride of the cat gallop, trot, and human run are plotted in Figs. 11 – 14. In each case, the data are in agreement with the even and odd symmetries predicted by the symmetry equations. The cat data for galloping show a remarkable degree of symmetry.

These running symmetries can be visualized graphically. The symmetry equations imply that if we reverse both the direction of forward travel and the direction of time, $x = -x$, $t = -t$, then the pattern of forward body movement and of footfalls should not be

Fig. 15. Pattern of foot contacts in the rotary gallop of a cat. Horizontal bars indicate that the foot is in contact with the support surface. The duration of an entire stride is 350 ms. Vertical dotted lines indicate the seven frames used in Fig. 16.



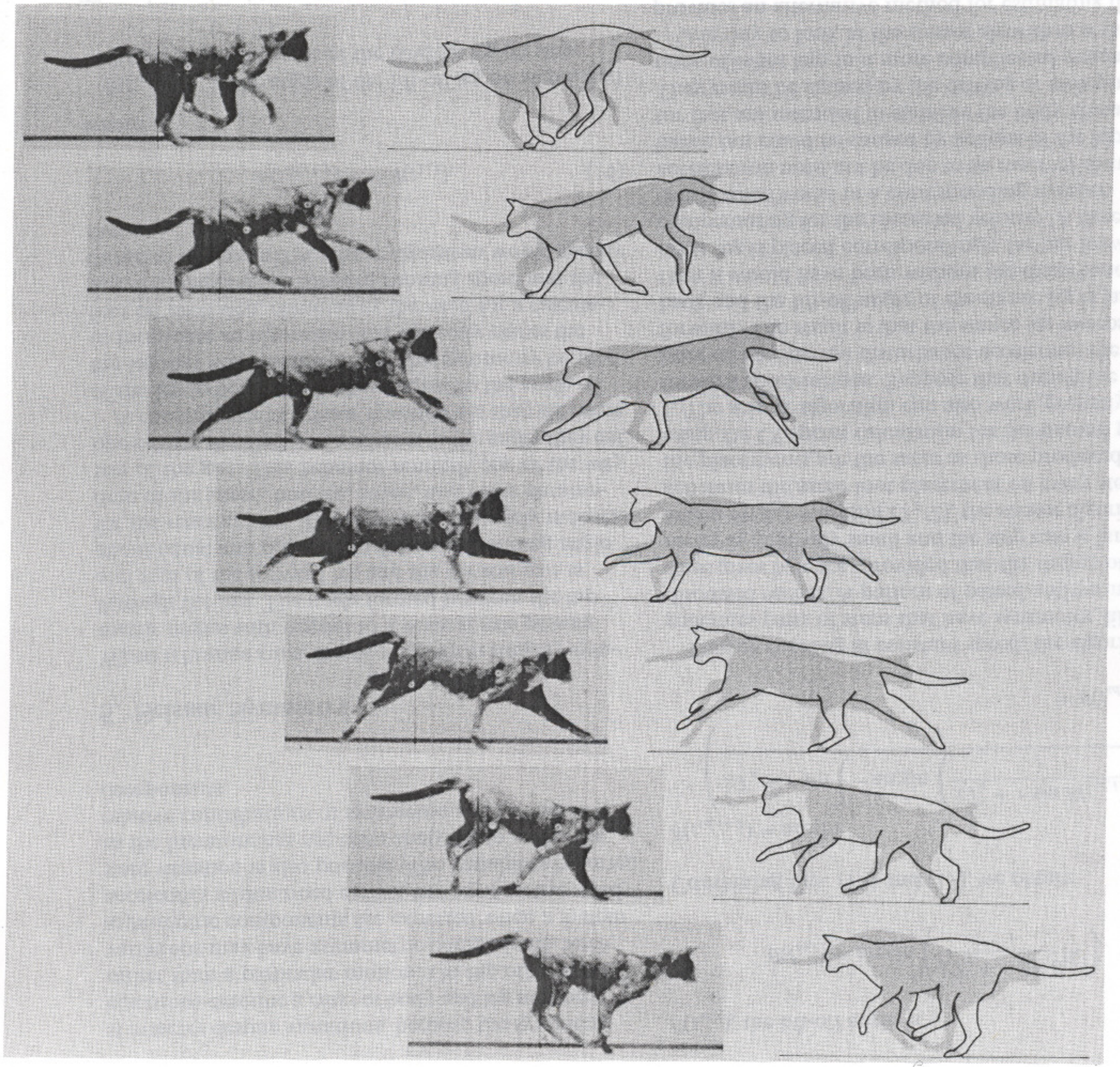
affected: $x(t) = -x(-t)$. This invariance is illustrated in Fig. 16. Of particular interest is the precise overlap of the footfalls for the forward and reverse running sequences. This overlap was predicted by the symmetry equations.

Some of the data examined reveal a bias in foot position toward the rear of the animal. The timing diagram of Fig. 15 illustrates such a bias. For each leg, $|\theta(t_{lo})| > |\theta(t_{td})|$, and the last leg providing support to the body stayed in contact with the ground longer than the first leg providing support. According to the principles outlined in this paper, such bias or skew might mean that a net forward force was generated on the body. Such a force could accelerate the system forward, compensate for an external disturbance, or compensate for losses occurring elsewhere in the system.

Another explanation, however, might be that the axial leg force does not obey Eq. (15). For instance, because the legs are not massless, their collisions with the ground on each step result in asymmetric forces. One might also expect the legs to deliver thrust actively during support, in order to make up for losses and to maintain the vertical bouncing motion. This active thrusting would result in a violation of Eq. (15). Without knowing the actual force that each leg exerts on the ground, it is difficult to draw definite conclusions regarding the implication of the observed asymmetry in foot position.

Animals do not run with a pattern of motion that is precisely repeatable from one stride to the next, even for a single gait. This variability has been reported in studies of interlimb coordination in the cat: for example see (Stuart et al. 1973; Miller, van der Burg, and van der Meché 1975; English 1979; and Vilensky and Patrick 1984). In principle, variability need not influence symmetry, and the two may be orthogonal. A legged system can switch from one symmetric pattern of motion on one stride to a different symmetric pattern on the next stride. A legged system can maintain

Fig. 16. Graphical interpretation of symmetry in the galloping cat. (left) *Photographs of a galloping cat taken at 50-ms intervals.* (right) *The shaded figures show the forward translation and the configuration of the cat during normal running, and the outlines show reverse running. The outlines were made from the same photographs as the shaded figures, but were reflected about the vertical axis and are presented in reverse sequential order,* $x(t) = -x(-t)$. *Therefore the outline at the top was made from the photograph at the bottom after reversing its orientation. The positions of supporting feet and the rightward motion of the body correspond quite well in the two sequences, as predicted by symmetry.* (diagram construction) *The relative placement of the figures for each sequence—the photographs, the shaded silhouettes, and the outlines—accurately reflects the forward progress of the cat with respect to the surface of the treadmill. After each set of figures was assembled according to the forward travel, the three sets were positioned relative to one another. Photographs are from film provided by Wetzel, Atwater, and Stuart (1976).*



symmetry despite variations, because the symmetry equations describe a *class* of body and leg motions rather than a particular motion. On the other hand, variations may have asymmetric components. Such asymmetric components are expected when a system accelerates as described earlier, but asymmetric steady-state variation is also possible. The variability reported in the literature has not been analyzed to reveal the relative contributions of symmetric and asymmetric components.

5. Scissor Symmetry

When a human runs, the two legs form roughly symmetric angles with respect to a vertical axis passing through the hip. The angle formed between the hip and foot of the forward leg and the vertical axis is about equal and opposite to the corresponding angle for the rearward leg. This symmetry is largely independent of the speed, bounce, stride, and other parameters of the gait. This behavior reminds one of the way one orients the blades of a scissors to the paper they cut.

A consequence of scissor symmetry is that the angle of the leg to be placed is about equal to the angle of the leg that was just lifted. Can this symmetry be used to formulate an algorithm that correctly places the foot on each step, eliminating the need for a calculation that depends on forward running speed and the duration of support? A scissor algorithm would specify that

$$\theta(t_{td,i+1}) = -\theta(t_{lo,i}), \tag{16}$$

where

- $\theta(t_{lo,i})$ is the angle of the lift-off leg on step i, and
- $\theta(t_{td,i+1})$ is the angle of the landing leg on step $(i+1)$.

The scissor algorithm of Eq. (16) could be used to specify foot placement for systems with any number of legs, provided that the gait used only one leg for support at a time. What sort of behavior would result?

When running with constant forward speed $\dot{x}$ and uniform stance duration T_s, the foot moves a distance $\dot{x}T_s$ backward with respect to the hip during the support interval. For a given landing angle of the leg $\theta(t_{td,i})$, the lift-off angle is

$$\theta(t_{lo,i}) = \arcsin\left(\frac{\dot{x}T_s + r\sin\theta_{td,i}}{r}\right). \tag{17}$$

Combining Eqs. (16) and (17), we obtain

$$\theta(t_{td,i+2}) = \arcsin\left(\frac{\dot{x}T_s + r\sin\left(-\arcsin\left(\frac{\dot{x}T_s + r\sin\theta(t_{td,i})}{r}\right)\right)}{r}\right) = \theta(t_{td,i}). \tag{18}$$

During running at constant speed, the algorithm generates pairs of steps that have symmetry, like those discussed earlier. A pattern of paired antisymmetric steps gives balance, provided that the degree of asymmetry is relatively small and the step rate is large. When $\theta(t_{lo,i}) = \arcsin(\dot{x}T_s/2)$, the scissor algorithm generates the same foot placement on every step, and the placements are the same as those produced by using the CG-print calculation for the neutral point.

The scissor algorithm can also work properly during forward accelerations. Suppose that during the support interval an external disturbance accelerates the system forward. The result is that the stance leg sweeps farther back and the lift-off angle of the stance leg is larger than it would have been without the disturbance. The other leg is placed correspondingly further forward, compensating for the increased velocity. A decelerating disturbance works in a corresponding manner. The acceleration need not be due to an external disturbance but could be caused by actions of the hip actuator that are intended to stabilize the body attitude. They might be caused by the driving or swinging actions of other legs in a more complicated system.

One way to look at the scissor algorithm is that it provides an alternative method for estimating the length of the CG-print. The lift-off angle of the leg serves to indicate both the forward velocity and ground time. The faster the body moves forward relative to the ground, the further backward the foot moves during stance. The foot also moves backward further when the system spends more time on the ground. Therefore the angle of the leg at lift-off is determined by the product of the average forward velocity and the duration of stance. The scissor algorithm is attractive because it is difficult to estimate the length of the CG-print accurately. It avoids the need to measure explicitly the forward velocity of the body and the duration of stance.

There are several difficulties with the scissor algorithm. First, the leg angle at touchdown is also influenced by the leg angle at lift-off. The product of average velocity and ground time relates only the change in leg angle, so the starting angle of the leg at touchdown determines where it is at lift-off. In principle, the algorithm can generate a sequence of uniform symmetric steps. In practice, there is no mechanism to keep from drifting to antisymmetric pairs of skewed steps with diverging skew.

This problem might be overcome by somehow damping the foot placement excursions or by using information from previous steps to filter the two-step oscillations. Another alternative might be to take both the touchdown and lift-off angles into account when calculating the next foot placement:

$$\theta(t_{td,i+1}) = \frac{\theta(t_{td,i}) - \theta(t_{lo,i})}{2}. \tag{19}$$

Another problem with the scissor algorithm is that it may not be responsive to sudden changes in the forward speed of the body. The forward speed that determines the next foot placement is the average from the entire previous support interval. The latency inherent in this indirect measurement could result in sluggish response to disturbances.

5.1. Asymmetry in Running

Despite the value of symmetry, there are several reasons why one should not expect to see perfect symmetry in the behavior of legged machines and animals. One reason for asymmetry is that legs are not lossless. The arguments used to motivate the relationship between symmetric motion and steady-state behavior do not apply in the presence of friction. In particular, the behavior of the system moving forward in time is no longer symmetric to its behavior moving backward in time. The details of the discrepancy depend on the details of the losses and on the geometry of the system. Another energy loss contributing to asymmetric motion is due to unsprung mass in the legs. Each time a foot strikes or leaves the ground, the system loses a fraction of its kinetic energy. In order to maintain stable locomotion, the control system must resupply energy on each cycle to compensate for these losses. For instance, the leg lengthens during the support interval and shortens during flight to maintain a stable hopping height. This can be done only by delivering asymmetric forces and torques through the actuators.

Another reason for asymmetric behavior is asymmetry in the mechanical system. Most animals have large heavy heads at one end of their bodies that are not counterbalanced by large heavy tails at the other end. Front and rear legs often vary in size, and the hips and shoulders may not be equally spaced about the center of mass. Each of these factors may induce asymmetry in the motions that can provide balanced, steady-state behavior. This is less of a problem for laboratory machines because they can be designed to conform to whatever mechanical symmetry is required.

Naturally, we shouldn't expect to see symmetric motion when the control system purposely skews the motion to change running speed. In this case, asymmetry in the motion provides the forces that accelerate the body. An external load, such as that produced by wind resistance or a draw-bar load, would also require a component of asymmetry in the motion of the body and legs. A runner at the start of a footrace and the driver of a jinrikisha demonstrate these sorts of asymmetric behavior.

Perhaps a better view is to think of locomotion in terms of the sum of a symmetric part and an asymmetric part. The symmetric part of the motion during each stride maintains steady-state behavior. Deviations from symmetry compensate for losses and provide acceleration.

The symmetry discussed in this paper postulates that each body variable, each leg variable, and each actuator variable has an even or odd symmetry. The net result of their interaction is to constrain the forces acting on the body throughout a stride so that they preserve the body's forward speed, elevation, and pitch angle. One might imagine a less complete symmetry that does not require symmetry of the basic variables individually but requires symmetry only in the net

Fig. 17. Two functions that integrate to zero. One is symmetric and one is not. Symmetry provides a sufficient condition but not a necessary condition for zero net forward acceleration.

forces and torques acting on the body:

$$f_x(t) = -f_x(-t),$$
$$f_y(t) = f_y(-t), \quad (20)$$
$$\tau_\phi(t) = -\tau_\phi(-t).$$

Stated alternatively, the body moves with symmetry while the legs do not. We have proved that this cannot be the case when only one leg is used for support at a time. The proof is given in Appendix B. However such solutions may be workable with additional legs.

6. What Does Symmetry Mean?

We can interpret symmetry in several ways. First, it is useful in the control of legged machines. The strategy used to control running machines was built around symmetry, and symmetry may play a role in achieving more complicated running behavior in the future. For instance, reciprocating leg symmetry is important in making a quadruped gallop.

Symmetry also helps us to characterize and understand the behavior we observe in animals. The analysis of symmetry in the cat and human shows that it describes how animals move when they trot, gallop, and run, and we expect to find that the same symmetries describe the motions of other animals running with other gaits. Perhaps most important is the idea that symmetry and balance give us tools for dealing with a dynamic system without requiring detailed solutions to complex formulations. Symmetry implies that each motion has two parts with opposing effects, just as balance requires equal and compensating forces and torques.

In certain respects, these symmetries are limited. They do not specify the details of a particular body motion that provides locomotion but merely give a broad classification that embodies several interesting features of the motion. The symmetries provide only sufficient conditions for successful locomotion, not necessary conditions (see Fig. 17). As far as we have been able to determine, the behavior of a legged system may violate the motion symmetries we have described with impunity, without limiting its ability to run and balance. Finally, these symmetries do not yield a specific prescription for control. They suggest only how the system should ultimately move and hint at possible avenues of attack.

In other respects, the symmetries described here are quite powerful. Three simple equations outline plausible body motions for systems with any number of legs engaged in a wide variety of gaits. Another small set of equations describes how the legs move. Although the symmetries do not specify individual motions or how to produce them, they provide rules that govern a large class of successful motions and suggest a wide variety of experiments.

This work on symmetry falls into a broader context that splits responsibility for control between the control system and the mechanical system being controlled. In this context, the control of locomotion is a low bandwidth activity that takes advantage of the intrinsic properties of the mechanical system. Rather than use a high bandwidth servo to move each joint of the legged system along a prescribed trajectory at high rate, the control system makes adjustments just once per stride. Once the foot has been positioned on each step, the mechanical system passively determines the details of the motion for the remainder of the stride. This approach depends on having a passive nominal motion that is close to the desired behavior. In the present context, symmetry is the means of achieving the nominal motion. This sort of approach may have value only for systems that perform repetitive behaviors. For instance, aside from juggling and handwriting (Hollerbach 1980), robot manipulation may be unsuited to this approach.

7. Summary

Symmetric motions of the body in space and of the feet with respect to the body provide nominal motions for steady-state locomotion. A control system for running can produce steady-state behavior by choosing motions of the legs that give $x(t)$ and $\phi(t)$ odd symmetry and $z(t)$ even symmetry. The leg motions chosen are themselves described by odd and even symmetries. This method applies to a number of legged configurations and helps to describe the behavior of running animals.

The significance of these symmetric motions is that they permit a control system to manipulate the symmetry and skewness of the motion, rather than the detailed shape of the motion. When the system's behavior conforms to Eqs (1 – 3), all forces acting on the body integrate to zero throughout one stride, so the body experiences no net acceleration. When behavior deviates from symmetry, the net acceleration of the system deviates from zero in a manageable way. The control task becomes one of manipulating these deviations.

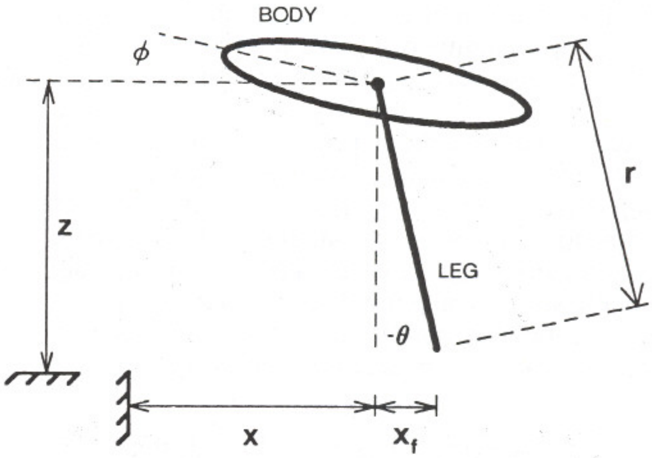
The conditions for symmetric body motion can be stated simply: at a single point in time during the support period, the center of support must be located under the center of mass, the pitch angle of the body must be zero, and the vertical velocity of the body must be zero, i.e., $\theta_j(0) + \theta_k(0) = 0$, $\phi(0) = 0$, and $\dot{z}(0) = 0$. The body follows a symmetric trajectory during stance when these conditions are satisfied.

Symmetric running motions may have great generality. In principle, a wide variety of natural running gaits can be achieved using body and leg motions that exhibit the symmetries described. These include the trot, the pace, the canter, the gallop, the bound, and the pronk, as well as the intermediate forms of these gaits. Although we have plotted symmetry data only for the cat and the human, we expect to find a wide variety of natural legged systems using nearly symmetric motions when they run.

Acknowledgments

I am grateful to Mary Wetzel for providing me with the cat film, to Jessica Hodgins and Jeff Koechling for helping film the human runners, and to Jeff Koechling for writing the computer programs used to digitize and process data.

Fig. A1. *Model of planar, one-legged system with a massless leg.*



Appendix A. Equations of Motion for Planar Systems

A.1. Equations of Motion for a Planar, One-Legged System

The equations of motion for a planar one-legged model, with a massless leg and the hip located at the center of mass (as shown in Fig. A1), are:

$$m\ddot{x} = f \sin\theta - \frac{\tau}{r}\cos\theta, \tag{A1}$$

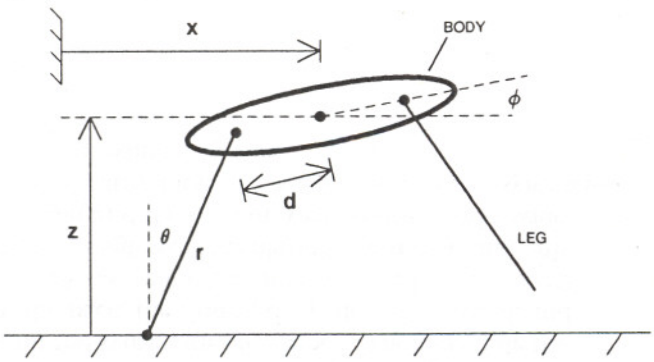
$$m\ddot{z} = f \cos\theta + \frac{\tau}{r}\sin\theta - mg, \tag{A2}$$

$$J\ddot{\phi} = \tau, \tag{A3}$$

where

x, z, ϕ	are the horizontal, vertical, and angular positions of the body,
r, θ	are the length and orientation of the leg,
τ	is the hip torque (positive τ accelerates body in the positive ϕ direction),
f	is the axial leg force (positive f accelerates the body away from the ground),
m	is the body mass,
J	is the body moment of inertia, and
g	is the acceleration of gravity.

Fig. A2. Model of planar, two-legged system with separated hips. It can represent the lateral half of a quadruped projected onto the sagittal plane or a biped projected onto the frontal plane.



A.2. Equations of Motion for a Planar, Two-Legged System

The equations of motion for planar model with two massless legs and hips located a distance d from the body's center of mass (as shown in Fig. A2) are:

$$m\ddot{x} = f_1 \sin \theta_1 + f_2 \sin \theta_2 - \frac{\tau_1}{r_1} \cos \theta_1 - \frac{\tau_2}{r_2} \cos \theta_2, \quad \text{(A4)}$$

$$m\ddot{z} = f_1 \cos \theta_1 + f_2 \cos \theta_2 + \frac{\tau_1}{r_1} \sin \theta_1 + \frac{\tau_2}{r_2} \sin \theta_2 - mg, \quad \text{(A5)}$$

$$J\ddot{\phi} = f_1 d \cos (\theta_1 - \phi) - \frac{\tau_1 d}{r_1} \sin (\phi - \theta_1) + \tau_1 - f_2 d \cos (\theta_2 - \phi) + \frac{\tau_2 d}{r_2} \sin (\phi - \theta_2) + \tau_2. \quad \text{(A6)}$$

Appendix B. Proof of Symmetric Leg Motion

In this appendix we prove that symmetric body motion requires symmetric leg motion for the one-legged case. Body and leg motions are symmetric when $x(t)$, $\phi(t)$, and $\theta(t)$ are odd and $z(t)$ and $f(t)$ are even.

The equations of motion can be rewritten expressing each element of the leg motion as the sum of an even and odd part. For instance, the angle of the leg with respect to the vertical is $\theta = {}^e\theta + {}^o\theta$, where ${}^e\theta$ represents the even part and ${}^o\theta$ represents the odd part. Also, r can be replaced with $1/({}^ez + {}^oz)$:

$$m\ddot{x} = ({}^ef + {}^of) \sin ({}^e\theta + {}^o\theta) - \tau({}^ez + {}^oz) \cos ({}^e\theta + {}^o\theta), \quad \text{(B1)}$$

$$m\ddot{z} = ({}^ef + {}^of) \cos ({}^e\theta + {}^o\theta) + \tau({}^ez + {}^oz) \sin ({}^e\theta + {}^o\theta) - mg, \quad \text{(B2)}$$

$$J\ddot{\phi} = {}^o\tau + {}^e\tau. \quad \text{(B3)}$$

From Eq. (B3), τ must be odd, and because ϕ is odd, we assume that $\ddot{\phi}$ is odd. To specify that the body moves with the desired symmetry, the even part of the right-hand side of Eq. (B1) is set to zero, and the odd part of the right-hand side of Eq. (B2) is also set to zero:

$$0 = {}^ef \sin {}^e\theta \cos {}^o\theta + {}^of \cos {}^e\theta \sin {}^o\theta + \tau\, {}^ez \sin {}^e\theta \sin {}^o\theta - \tau\, {}^oz \cos {}^e\theta \cos {}^o\theta. \quad \text{(B4)}$$

$$0 = -{}^ef \sin {}^e\theta \sin {}^o\theta + {}^of \cos {}^e\theta \cos {}^o\theta, + \tau\, {}^ez \sin {}^e\theta \cos {}^o\theta + \tau\, {}^oz \cos {}^e\theta \sin {}^o\theta. \quad \text{(B5)}$$

Solutions to Eqs. (B4) and (B5) require that

$$\tan {}^e\theta = \frac{\tau\, {}^oz}{{}^ef} \quad \text{and} \quad \tan {}^e\theta = -\frac{{}^of}{\tau\, {}^ez}. \quad \text{(B6)}$$

During the support interval, the foot remains stationary with respect to the ground, so motion of the body with respect to the ground determines motion of the foot with respect to the body. Therefore the symmetries of Eq. (1) and the solutions to Eq. (B6) also govern the trajectory of the foot with respect to the body. They require that $x_f(t) - x_f(0) = -x_f(-t) + x_f(0)$ and $z_f(t) = z_f(-t)$. The leg motion is symmetric if $x_f(0) = 0$.

Because odd functions equal zero when $t = 0$, Eq. (B6) requires that ${}^e\theta(t = 0) = 0$, implying that $x_f(0) = 0$. Hence ${}^e\theta = {}^or = {}^of = 0$, leaving θ odd and r and f even. They obey the leg symmetries given by Eq. (2).

Goldberg (1985) simplified the proof as follows. The foot remains stationary with respect to the ground during the support interval, so motion of the body with respect to the ground determines the motion of the foot with respect to the body. Therefore the symmetries of Eq. (2) govern the trajectory of the foot with respect to the body:

$$x_f(t) - x_f(0) = -x_f(-t) + x_f(0), \qquad \text{(B7)}$$

$$z_f(t) = z_f(-t). \qquad \text{(B8)}$$

The leg motion is symmetric if $x_f(0) = 0$.

Let f_x and f_z be the horizontal and vertical forces between the foot and the ground. The torque at the hip can be written as

$$\tau = -f_x z_f + f_z x_f. \qquad \text{(B9)}$$

From the equations of motion, we know that τ and f_x are odd and that f_z is even, so Eq. (B9) requires that x_f be odd. Therefore leg angle $\theta = \arctan(x_f/z_f)$ is odd and leg length $r = \sqrt{x_f^2 + z_f^2}$ is even. Axial leg force $f = (f_x x_f + f_z z_f)/r$, which is even.

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