One Dimensional Climber Notes

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1 Introduction

The one dimensional climber is an example developed between myself, Mark, and Dan Santos about what a really simple climber could look like.

The climber consists of a body connected to a mass-less foot via a spring, damper, and force actuator in parallel. The foot can clamp onto a vertical pole with a given clamping (normal) force, or release the pole. Gravity acts down the pole:

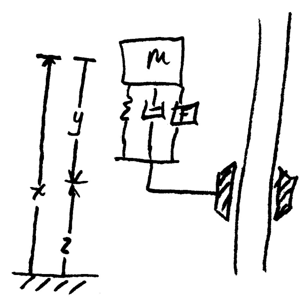

Figure 1: One-Dimensional Climber

The goal of this model is to provide a relatively simple dynamic platform that (hopefully) remains analytically tractable, while being interesting enough to exhibit behavior which may be classified as "dynamic climbing" by utilizing some form of sliding during climbing. The model may be used to answer "what if?" type questions, such as, "What if the friction model includes some form of viscous term?" Ultimately, the model could be used as a template for more complex climbing robots.

2 The Model

Friction is modeled using a Karnopp friction model when the climber has clamped the pole (see Figure 2). The static friction limit is slightly more than the weight of the climber ($F_s \leq mg + \delta$, $\delta > 0$), and the kinetic friction force generated when sliding is slightly less than the weight of the climber ($F_d = mg + \zeta$, $\zeta < 0$).

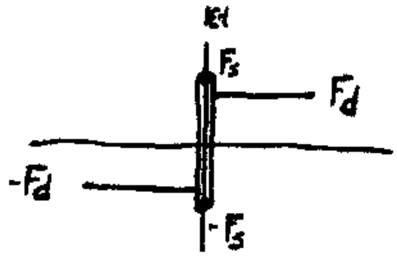

Figure 2: Karnopp Friction Model

The model uses the following variables:

- x - absolute coordinate of the body
- y - relative coordinate from body to foot
- y_n - uncompressed length of spring
- z - absolute coordinate of the foot, $z = x - y$
- r - amount of spring extension, $r = y - y_n$
- m - mass of climber
- g - gravity
- F - actuator force
- F_w - wall force applied to foot (friction force)
- k - spring constant
- b - damping constant
- ε - one-half of the Karnopp Friction deadband

There are four fundamental states in this system: Clamped, Free, Sliding Down, and Sliding Up. The transition into and out of the Free state is controlled by clamping or releasing the climber. Transition from the clamped state to sliding up and down occurs when the wall reaction force exceeds the static friction force limit. The climber stops sliding if the speed of the foot in relation with the pole falls below ε.

The equations of motion for each state can be found in the appendix.

3 Phase Portraits

In each state, the dynamics can be characterized by two variables, $\dot{x}$, the velocity of the mass, and r, the extension of the spring. The phase portrait is a representation of the dynamics for each state as a vector field. Each vector in the field depicts the state velocity at that point. State transitions are also depicted on each phase portrait - almost all are linear with a slope of k/b. For each state, there is a separate phase portrait for each level of applied force, F.

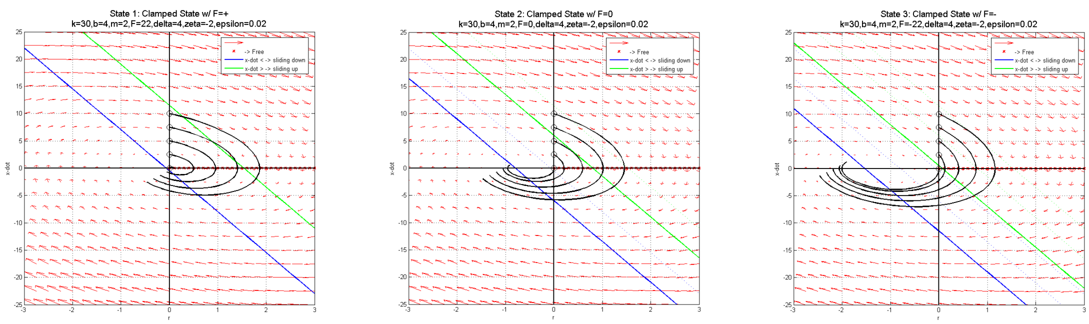

Figure 3: Clamped State Phase Portraits

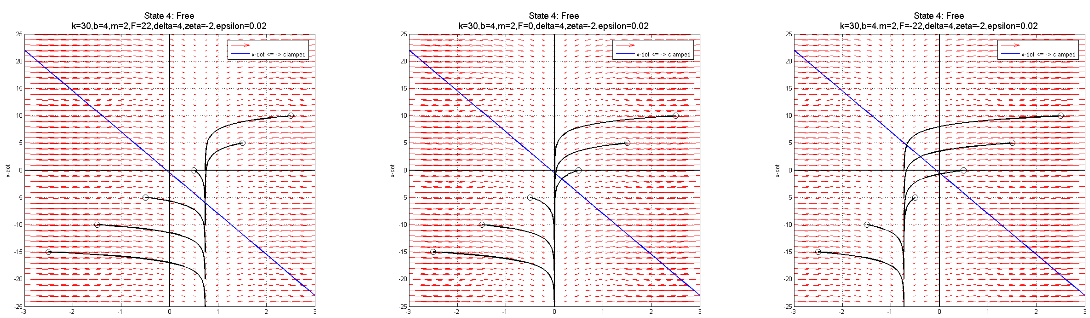

Figure 4: Free State Phase Portraits

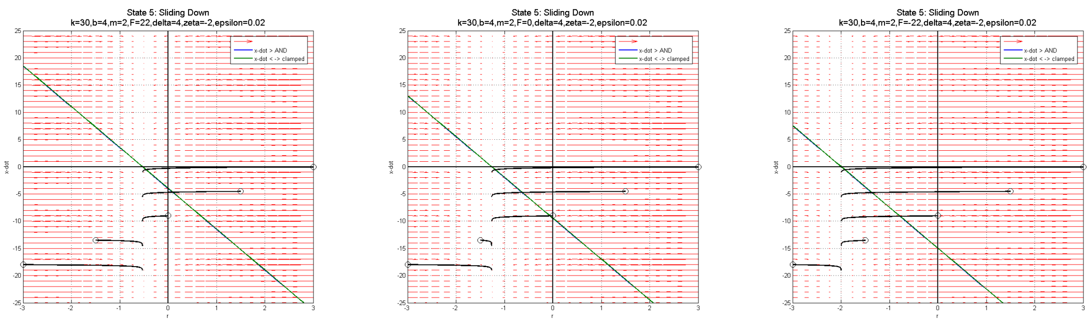

Figure 5: Sliding Down State Phase Portraits

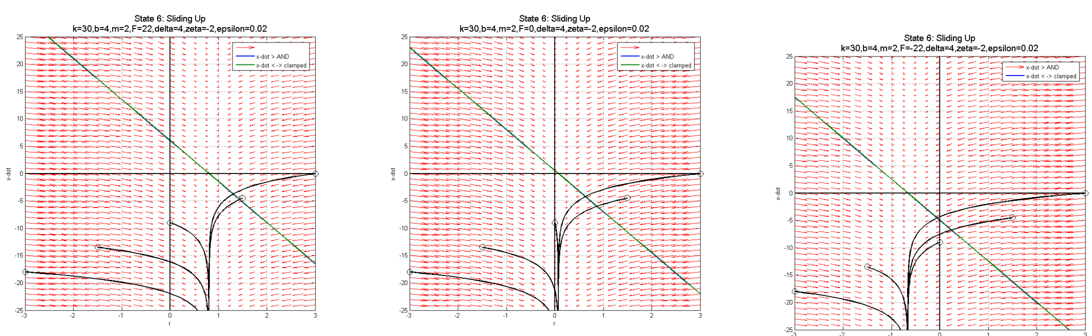

Figure 6: Sliding Up State Phase Portraits

4 Climbing without Slipping

In order to climb without slipping, the transition from free to clamped must occur such that the resulting trajectory does not exceed the static friction limit while the climber is in the clamped state.

A simple control strategy to accomplish this is to clamp the foot when the system is at the static friction limit, as shown by the blue line in Figure 3. To ensure that the resulting trajectory does not exceed the static friction limit, the state velocity at the transition point must point to the right of the static friction limit line. Mathematically: $\begin{bmatrix} r \\ \dot{x} \end{bmatrix}' \cdot \begin{bmatrix} k \\ b \end{bmatrix} \geq 0$.

Each level of applied force will have a specific point in the $(r, \dot{x})$ space that corresponds to $\begin{bmatrix} r \\ \dot{x} \end{bmatrix}' \cdot \begin{bmatrix} k \\ b \end{bmatrix} = 0$ along the static friction limit line. Solving for this point analytically:

$$\begin{bmatrix} r \\ \dot{x} \end{bmatrix}' \cdot \begin{bmatrix} k \\ b \end{bmatrix} = \left(\begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix} \cdot \begin{bmatrix} r \\ \dot{x} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{F}{m} - g \end{bmatrix} \right) \cdot \begin{bmatrix} k \\ b \end{bmatrix} \tag{1}$$

$$= \begin{bmatrix} \dot{x} \\ -\frac{k}{m} r - \frac{b}{m} \dot{x} + \frac{F}{m} - g \end{bmatrix} \cdot \begin{bmatrix} k \\ b \end{bmatrix} \tag{2}$$

$$= k\dot{x} - \frac{bkr}{m} - \frac{b^2\dot{x}}{m} + \frac{Fb}{m} - gb \tag{3}$$

In the clamped state, along the static friction limit, $F_w = F_s$, and $\dot{x}$ can be written as a function of r:

$$\dot{x} = \frac{-kr + F - F_s}{b} \tag{4}$$

Substituting Equation 4 into Equation 3 yields:

$$\frac{-k^2 r}{b} + \frac{kF}{b} - \frac{kF_s}{b} - \frac{bkr}{m} + \frac{bkr}{m} - \frac{bF}{m} + \frac{bF_s}{m} + \frac{bF}{m} - gb = 0 \tag{5}$$

$$\Rightarrow \frac{-k^2 r}{b} + \frac{kF}{b} - \frac{kF_s}{b} + \frac{bF_s}{m} - gb = 0 \tag{6}$$

$$\Rightarrow r = \frac{F - F_s}{k} + \frac{b^2 F_s}{mk^2} - \frac{gb^2}{k^2} \tag{7}$$

$$\Rightarrow \dot{x} = -\frac{F - F_s}{b} - \frac{bF_s}{mk} + \frac{gb}{k} + \frac{F - F_s}{b} \tag{8}$$

$$\Rightarrow \dot{x} = \frac{gb}{k} - \frac{bF_s}{mk} \tag{9}$$

The most interesting result is that $\dot{x}$ does not depend upon the applied force, F. Once the mass starts moving at $\dot{x}$, sliding is inevitable upon clamping if the climber clamps at the static limit line. Note that if there are no restrictions on the levels and resolution of F that can be applied, it is possible to prevent sliding after the mass velocity passes this speed limit by decreasing the applied force to a point such that the resulting trajectory does not exceed the static friction limit.

5 Climbing with Discrete Force Levels

If the force actuator is limited to applying a set of discrete amounts of force, the clamped state phase plots corresponding to the various forces can be combined to get an idea of the possible trajectories, as shown in Figure 7.

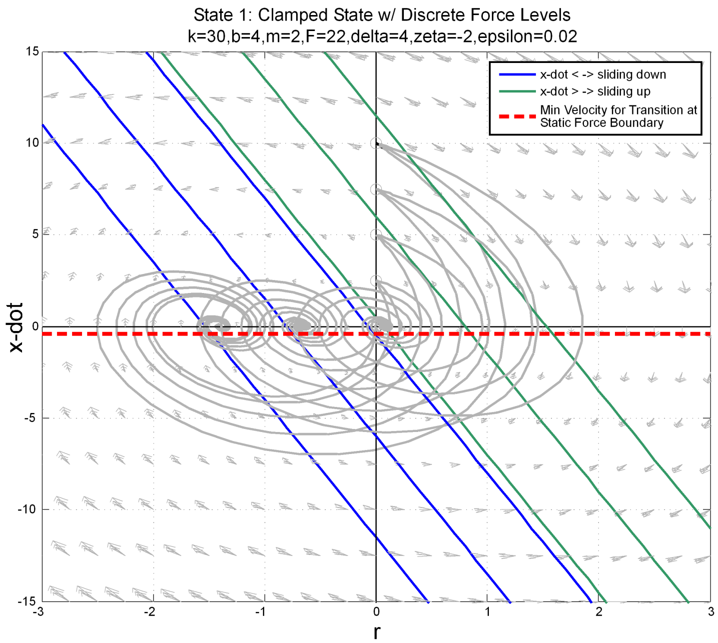

Figure 7: Phase Portrait with Multiple Possible Forces

An easy method of safely increasing the force while in the clamped state would be to require that the steady-state point for a certain force level lies within the static friction force limit of the next higher force. If this were the case, and the system could be brought to steady state at a lower force level, the force could then be increased to the next level, brought to the next steady state position, and then increased again. This is not possible for the force quantization shown in Figure 7, but is when the forces quantization is that in Figure 8 (example trajectory shown in orange). It can easily shown that this style of climbing is possible when the difference between subsequent force levels is less than δ.

Given a minimum (max negative) force that can be applied, the limit on $\dot{x}$ can be expanded in $(r,\dot{x})$ space to create a boundary which must not be crossed in the Free state if the climber is to prevent itself from sliding down the pole. This boundary is defined by integrating back in time from the point on the maximum static friction line where $\begin{bmatrix} r \\ \dot{x} \end{bmatrix}' \cdot \begin{bmatrix} k \\ b \end{bmatrix} = 0$. This boundary is sketched in red in Figure 8

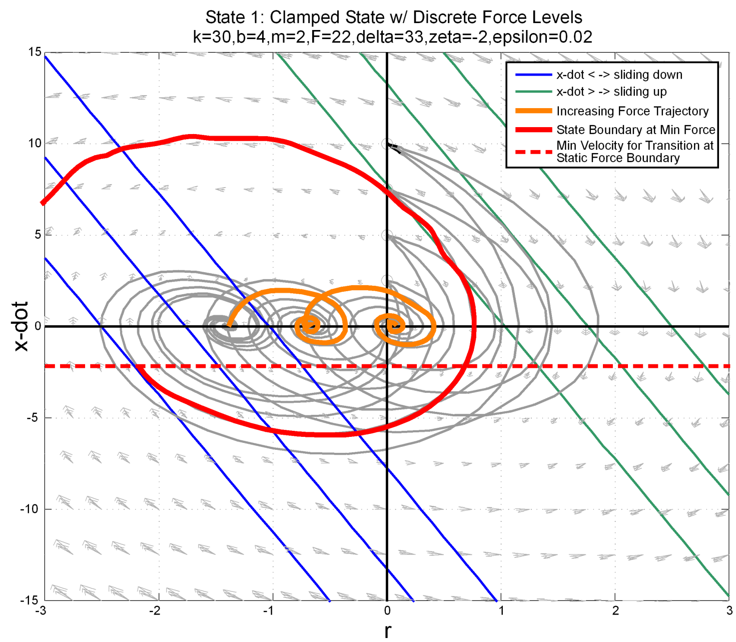

Figure 8: Phase Portrait with Multiple Possible Forces: Quantization $< \delta$

To be able to transition from free to clamped and then extend the spring to max extension without slipping on the pole, the transition from free to clamped must occur within the boundary trajectory for the desired force level (likely most negative), AND the resulting trajectory must cross the static friction force boundary for the next level of force, continuing until the maximum value of force is applied.

6 Transition from Free to Clamped

The requirements for the transition from free to clamped can be determined by plotting vector fields for the clamped states and free state on top of each other, as shown in Figure 9. Figure 9 is a nasty plot. There are three clamped phase plots, one for each of the three levels of force that I've assumed. The clamped phase plot velocity vectors and representative trajectories are depicted in light brown. The dark blue lines represent the transition to sliding down for the three different forces. The dark green lines are the transition to sliding up for the three different forces. The dark red lines represent the trajectory which you must be inside to clamp without sliding. The green lines are extensions of the boundary trajectories - these must cross the sliding boundary corresponding to the next force level for any hope of a closed trajectory.

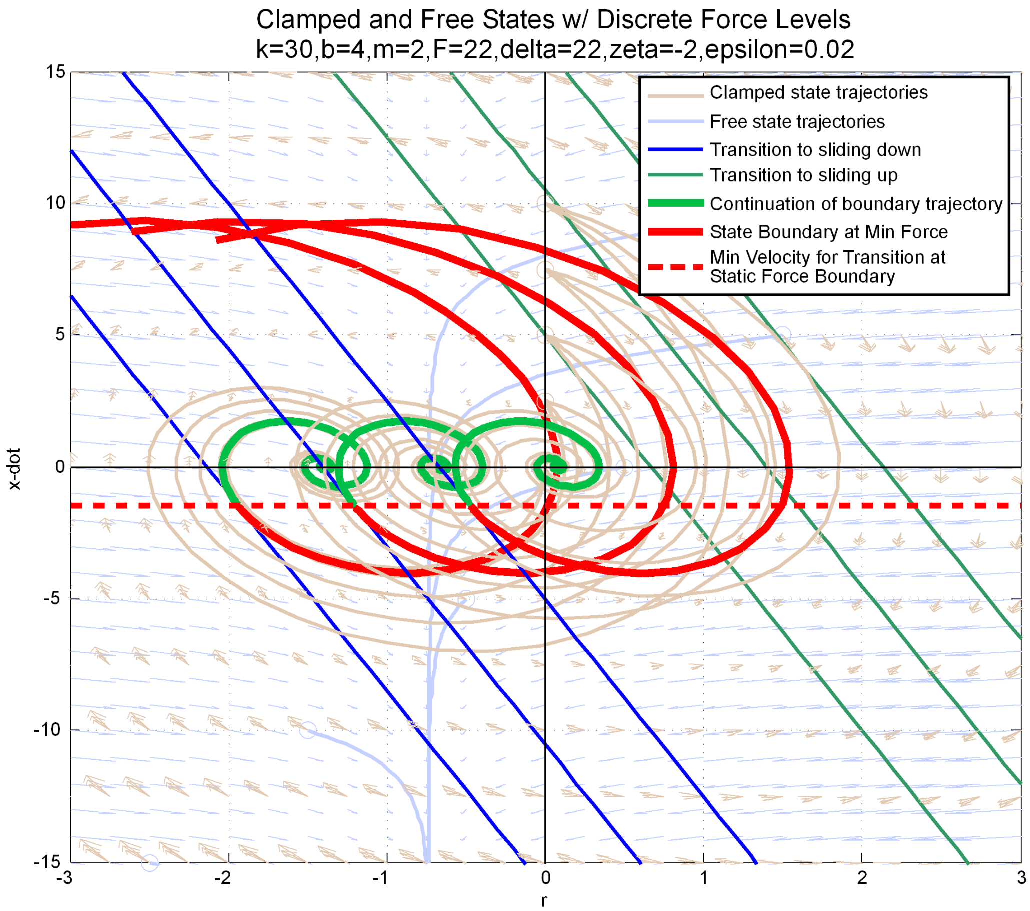

Figure 9: Clamped and Free Phase Portraits

Figure 9 indicates that a control strategy which transitions from free to clamped before departing the min force boundary, switching to the next force boundary once the slipping limit is passed for the next higher level of force, and then transitioning to free once the maximum force is reached and $\dot{x}$ crosses zero, will work, at least when $\Delta F < \delta$.

7 Transition from Sliding Down to Clamped

The transition from sliding down to clamped is tricky. The sliding down state only occurs when the required wall force to keep the foot stationary exceeds the limit of static friction, and the foot starts slipping down. If the slipping is sensed, and the force can be reduced, then the foot quickly comes back to zero velocity. There is some limit (assuming a minimum force) where the force cannot prevent the foot from slipping down and the system is unable to stop sliding down the pole. If a continuous range of forces is available (down to the minimum), than the system can just apply the required force, the foot stops sliding, and the system returns to the clamped state.

This all changes if a viscous term is added to the friction model.

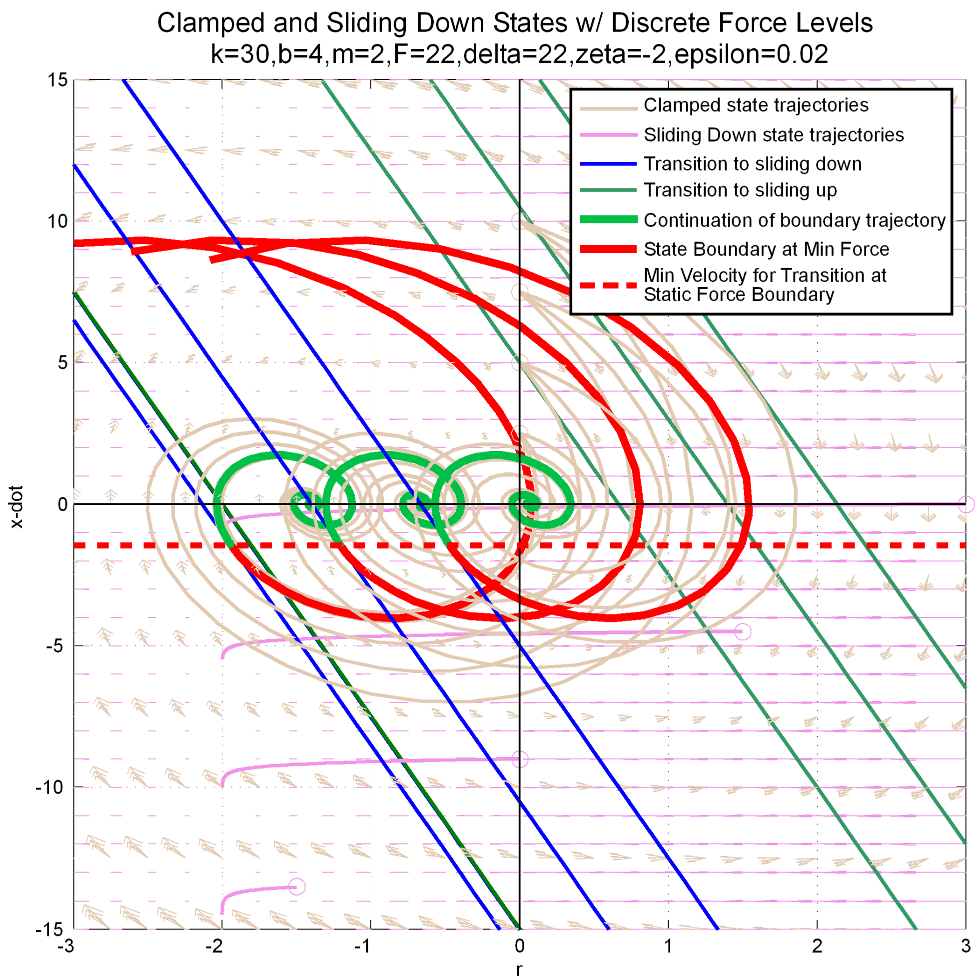

Figure 10: Clamped and Sliding Down Phase Portraits

Figure 10 indicates that there is a set of points in the state space

8 Sample "Dynamic" Trajectory

As an example, a trajectory was built using the following rules:

1. Transition from clamped to free when $\dot{x} > 0$ and $r = (F - mg)/k$
2. Apply $-F$ during free
3. Clamp at $\dot{x} = 0$
4. Step up force when you cross the sliding down boundary
5. Start at $(0, -mg/k)$ - the steady state position with 0 force, and apply F

This will work for a system where the force quantization is less than the difference between the limit of static friction force and the weight of the system.

(Side note: Dan K. was obviously sidetracked by my talking about force quantization - it's semi-interesting to me, but not that important overall. Talking about max and min force authority is probably all I need to do.)

This control strategy is implemented in simulation for a system with the following paramaters:

k = 30 N/m
b = 4 N-s/m
y_n = 0.05 m
m = 2 kg
g = 10 m/s^2
δ = $2mg/3$ N
ζ = $-mg/10$ N
ε = $mg/1000$ N
F = δ N

Key numbers in there are that $\delta > mg/2$ and $F = \delta$

Simulation yields the following phase plots and time histories:

9 Sample "Quasi-Static" Trajectory

A quasi-static example trajectory:

1. Transition from clamped to free when $\dot{x} >= 0.1$

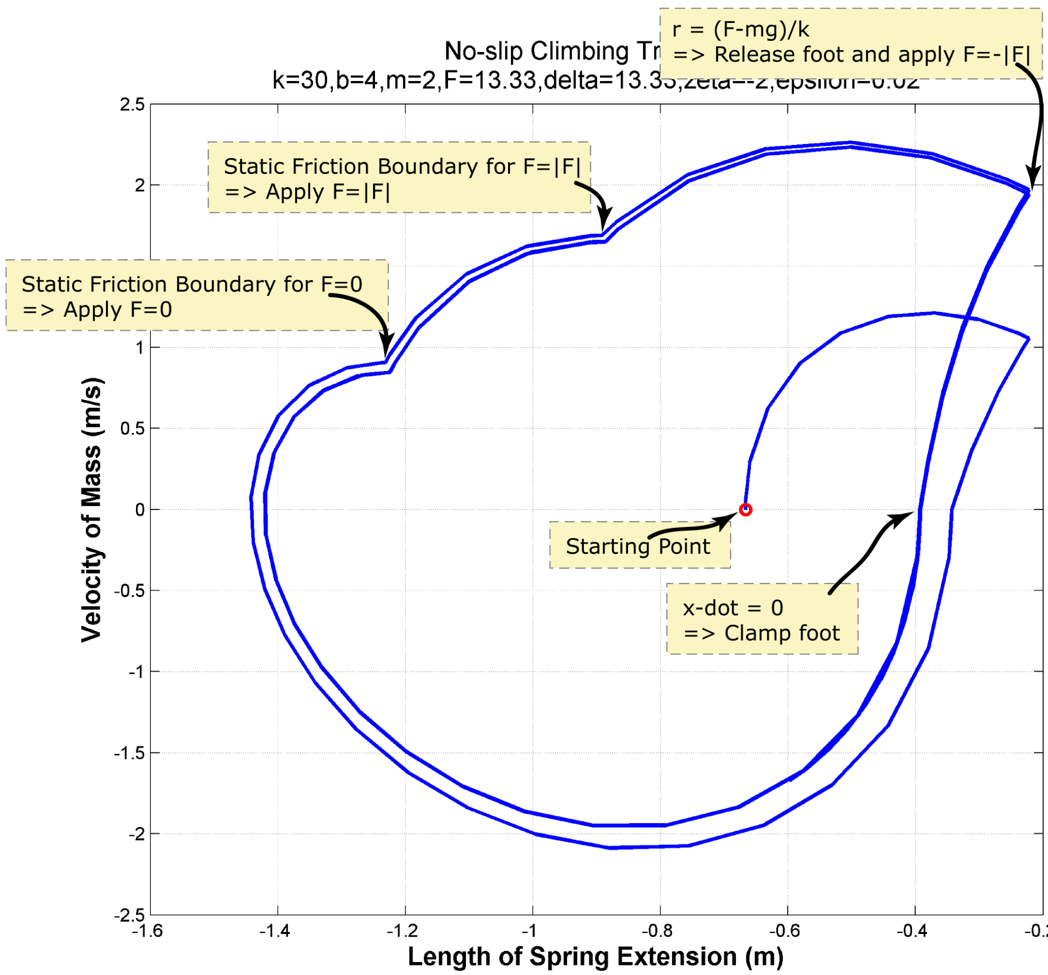

Figure 11: Sample Trajectory Phase Portrait

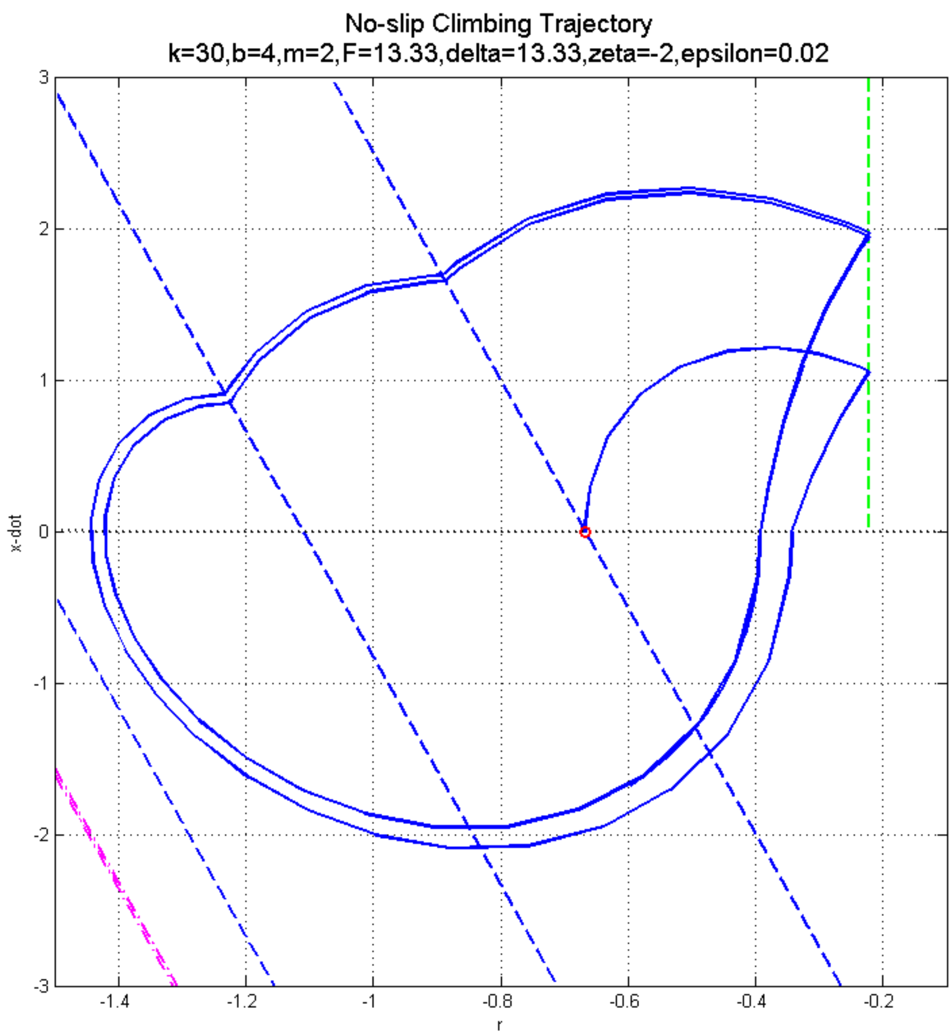

Figure 12: Sample trajectory phase portrait with slip boundaries

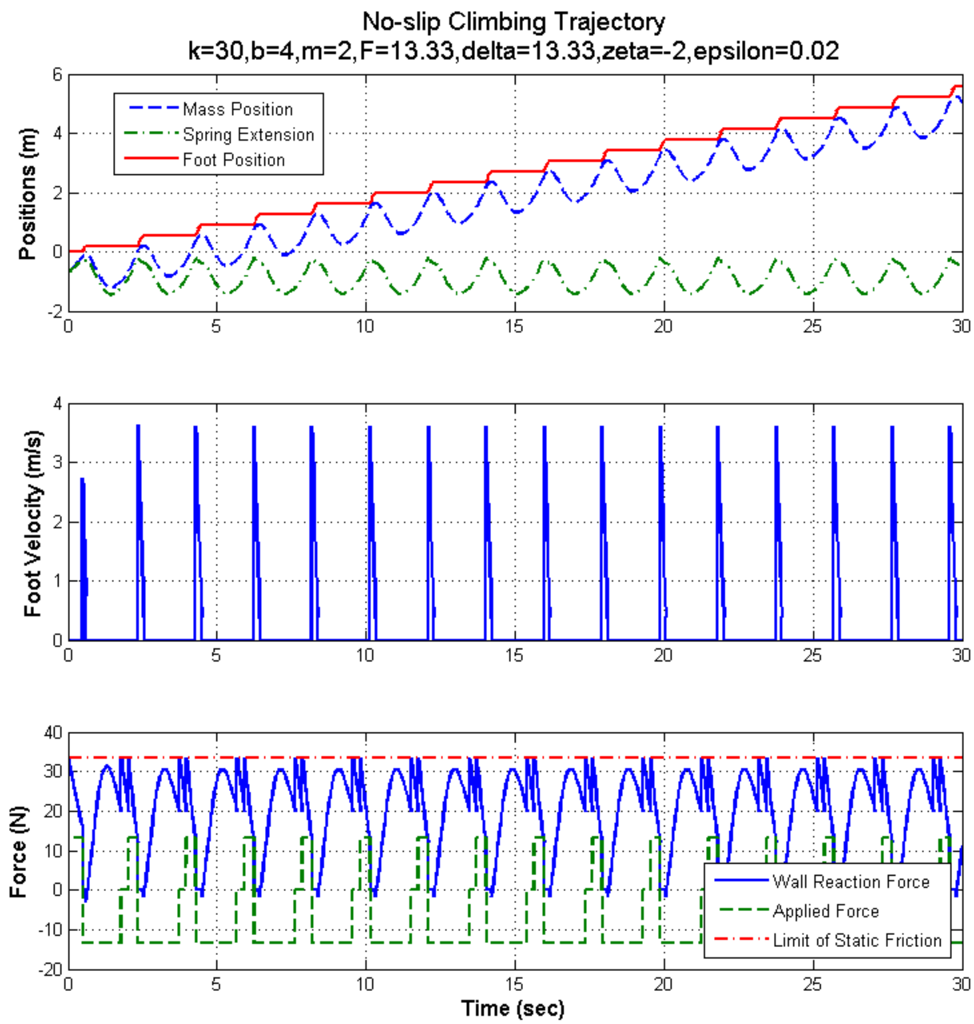

Figure 13: Sample Trajectory Time Histories

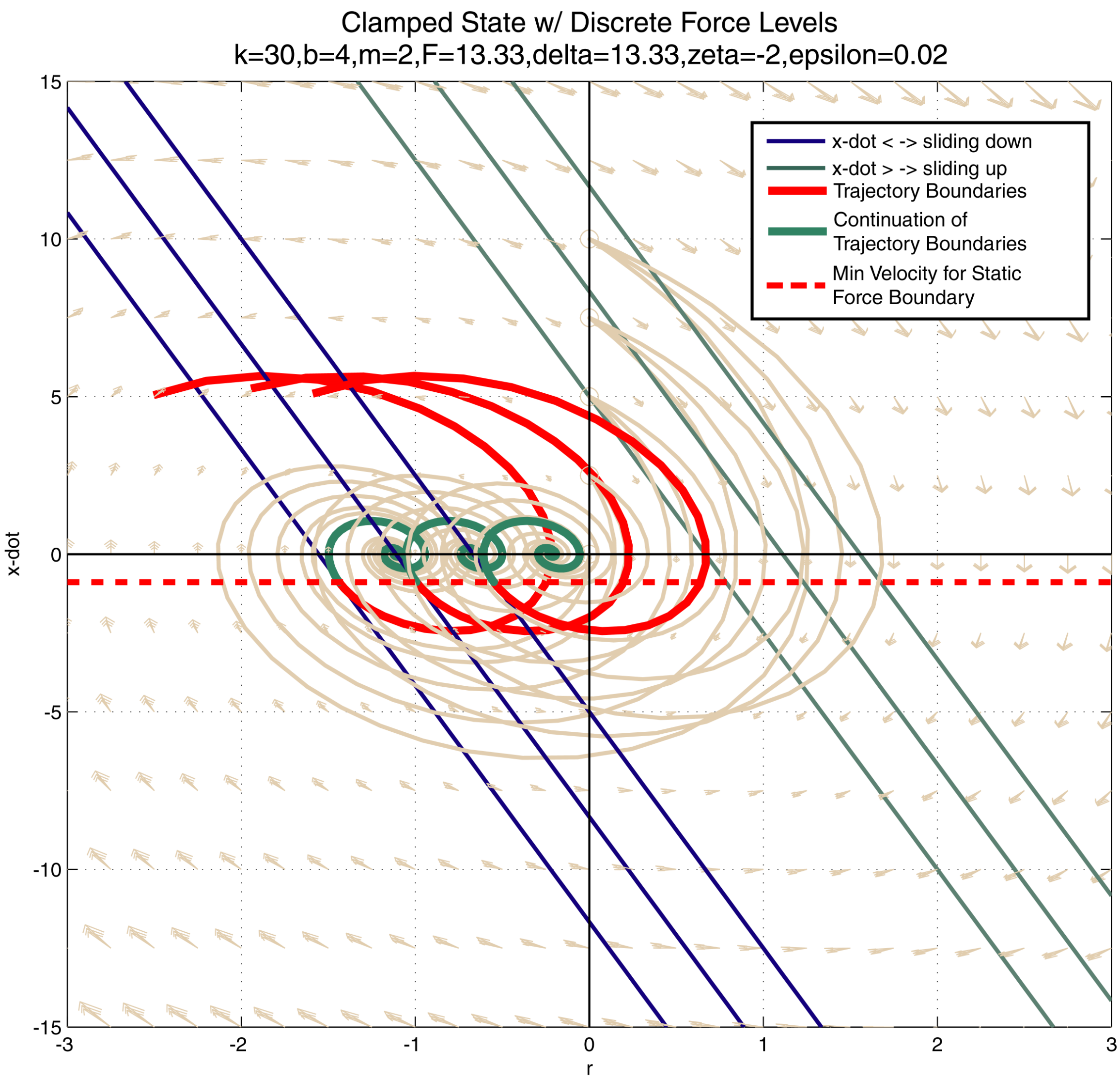

Figure 14: Sample Trajectory Clamped Phase Plot

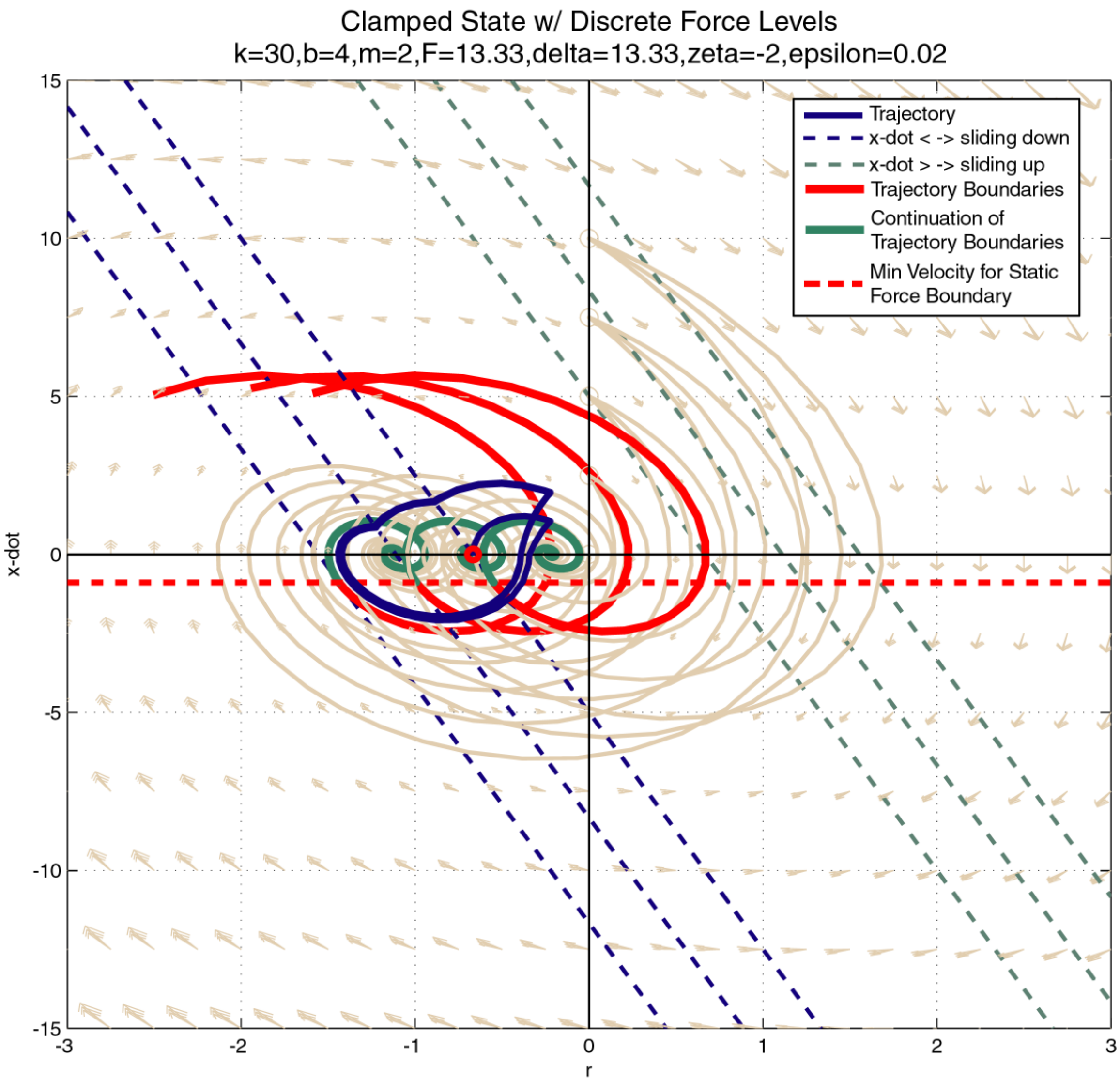

Figure 15: Sample trajectory clamped phase plot with actual trajectory

2. Apply $-F$ during free
3. Clamp at $\dot{x} = 0$ and apply F
4. Start at $(0, -mg/k)$ - the steady state position with 0 force, and apply F

This control strategy is implemented in simulation for a system with the following paramaters (same as previous section):

k = 30 N/m
b = 4 N-s/m
y_n = 0.05 m
m = 2 kg
g = 10 m/s^2
δ = $2mg/3$ N
ζ = $-mg/10$ N
ε = $mg/1000$ N
F = δ N

Again, key numbers in there are that $\delta > mg/2$ and $F = \delta$

Simulation yields the following phase plots and time histories:

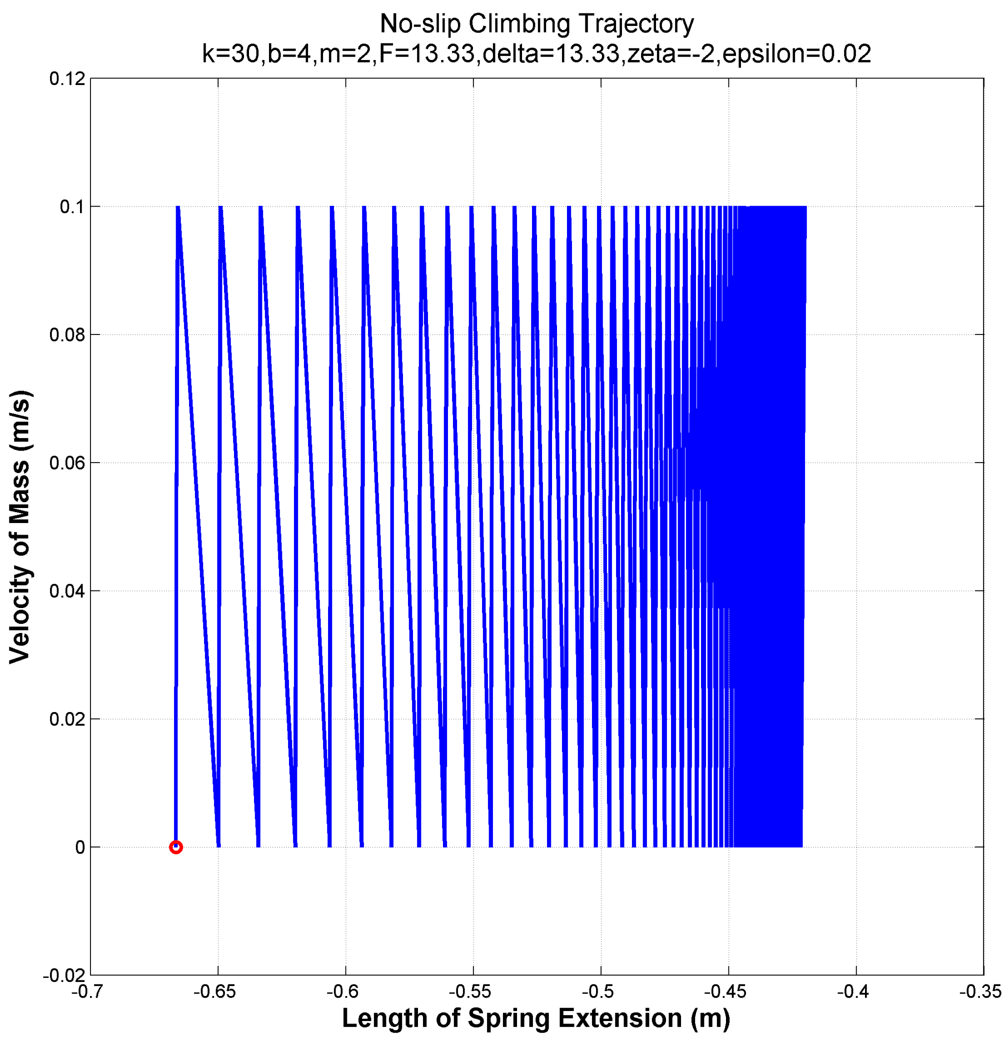

Figure 16: Sample Quasi-Static Trajectory Phase Portrait

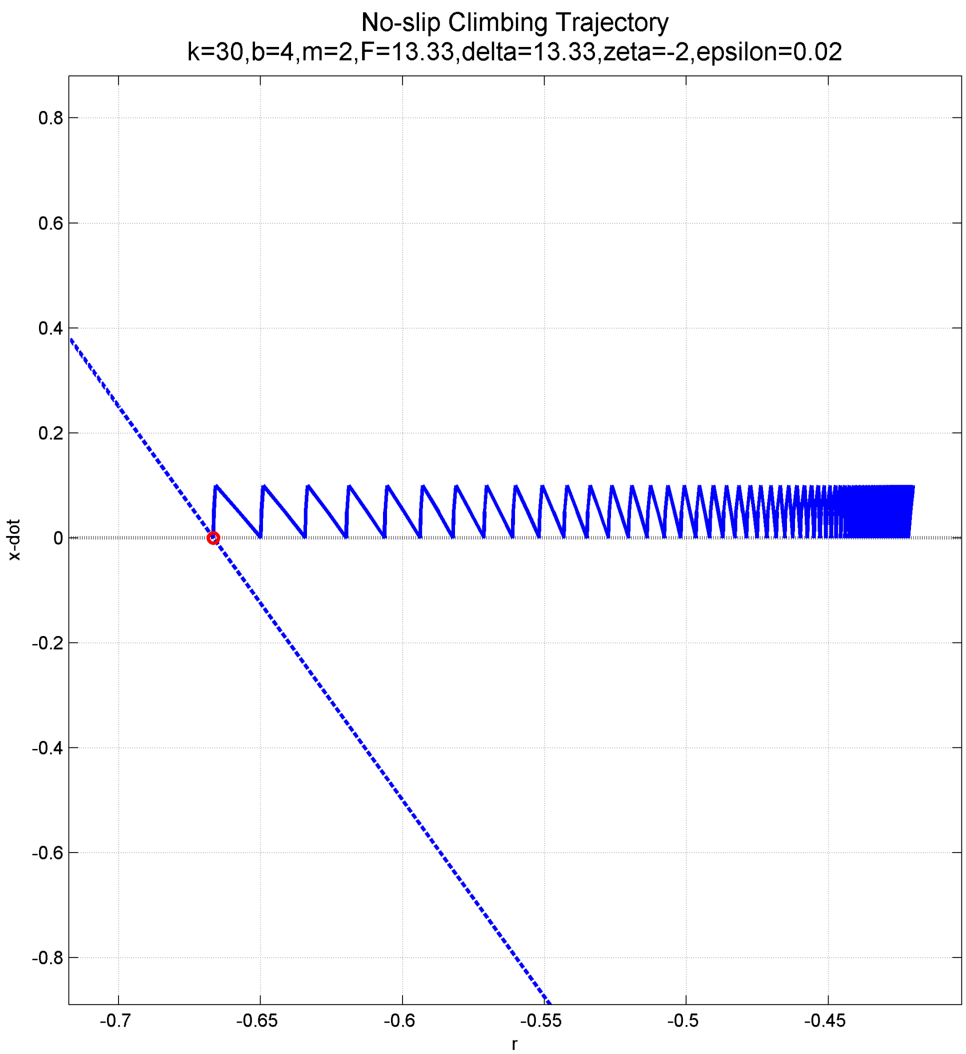

Figure 17: Sample quasi-static trajectory phase portrait with slip boundaries

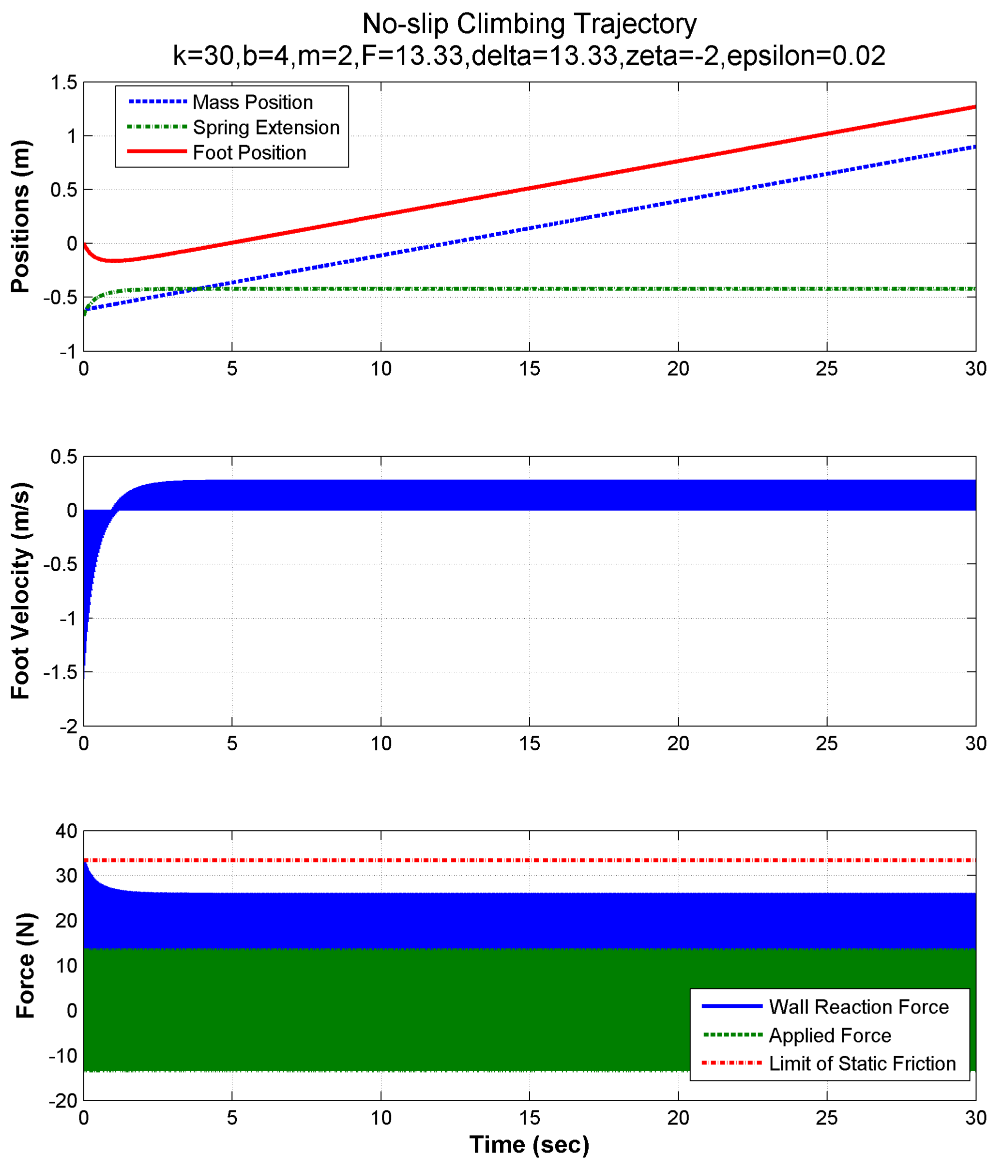

Figure 18: Sample Quasi-Static Trajectory Time Histories

A Appendix

Equations that don't change with state:

$$|F_{s_{\max}}| = mg + \delta \quad \text{limit of force due to static friction} \tag{10}$$
$$|F_d| = mg + \zeta \quad \text{dynamic friction force} \tag{11}$$
$$y = r + y_n \tag{12}$$

Equations when Clamped:

$$\dot{r} = \dot{x} \tag{13}$$
$$\dot{x} = \dot{x} \tag{14}$$
$$\ddot{x} = -\frac{k}{m}r - \frac{b}{m}\dot{x} + \frac{F}{m} - g \tag{15}$$
$$x = r \tag{16}$$
$$\dot{x} = \dot{x} \tag{17}$$
$$y = r + y_n \tag{18}$$
$$\dot{y} = \dot{x} \tag{19}$$
$$z = x - r - y_n \tag{20}$$
$$\dot{z} = 0 \tag{21}$$
$$F_w = -kr - b\dot{x} + F \tag{22}$$

Equations when Free:

$$\dot{r} = -\frac{k}{b}r + \frac{F}{b} \tag{23}$$
$$\dot{x} = \dot{x} \tag{24}$$
$$\ddot{x} = -g \tag{25}$$
$$x = x \tag{26}$$
$$\dot{x} = \dot{x} \tag{27}$$
$$y = r + y_n \tag{28}$$
$$\dot{y} = -\frac{k}{b}r + \frac{F}{b} \tag{29}$$
$$z = x - r - y_n \tag{30}$$
$$\dot{z} = \frac{k}{b}r + \dot{x} - \frac{F}{b} \tag{31}$$
$$F_w = 0 \tag{32}$$

Equations when Sliding Down:

$$\dot{r} = -\frac{k}{b}r + \frac{F - 2mg - \zeta}{b} \tag{33}$$
$$\dot{x} = \dot{x} \tag{34}$$
$$\ddot{x} = \frac{\zeta}{m} \tag{35}$$
$$x = x \tag{36}$$
$$\dot{x} = \dot{x} \tag{37}$$
$$y = r + y_n \tag{38}$$
$$\dot{y} = -\frac{k}{b}r + \frac{F - 2mg - \zeta}{b} \tag{39}$$
$$z = x - r - y_n \tag{40}$$
$$\dot{z} = \frac{k}{b}r + \dot{x} + \frac{2mg + \zeta - F}{b} \tag{41}$$
$$F_w = mg + \zeta \tag{42}$$

Equations when Sliding Up:

$$\dot{r} = -\frac{k}{b}r + \frac{F - \zeta}{b} \tag{43}$$
$$\dot{x} = \dot{x} \tag{44}$$
$$\ddot{x} = -2g - \frac{\zeta}{m} \tag{45}$$
$$x = x \tag{46}$$
$$\dot{x} = \dot{x} \tag{47}$$
$$y = r + y_n \tag{48}$$
$$\dot{y} = -\frac{k}{b}r + \frac{F - \zeta}{b} \tag{49}$$
$$z = x - r - y_n \tag{50}$$
$$\dot{z} = \frac{k}{b}r + \dot{x} + \frac{\zeta - F}{b} \tag{51}$$
$$F_w = -mg - \zeta \tag{52}$$