Limit Surface and Moment Function Descriptions of Planar Sliding

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Abstract

We present two geometric descriptions of the frictional properties of a rigid body sliding on a planar surface. The *limit surface* LS, from classical plasticity theory, is the boundary of the set of all possible frictional forces and moments that can be sustained by the frictional interface. Zhukovskii's *moment function* is the frictional moment as a function of the instantaneous center of rotation's location. Both of these descriptions implicitly contain the full relation between slip motion and frictional load for an object which makes contact governed by a useful class of friction laws which includes Coulomb friction. These surfaces can be used to deduce results concerning the overall frictional motion behavior of rigid bodies such as the existence of characteristic final slip motion directions.

Introduction

In order to effectively plan and control the motion of an object in contact with the ground and/or another object it is helpful to understand the nature of the friction contact forces between objects. However, much about the mechanics of frictional contact is still poorly understood. The micro-mechanisms responsible for friction include a variety of mechanisms (e.g. adhesion, plastic deformation, fracture) that conspire in a complex way to cause what we know macroscopically as friction, e.g. [11] Even the appropriate macroscopic descriptions for friction are not universally agreed upon. Possible dependencies of friction on normal force or stress, on normal separation distance, on slip displacement, on slip velocity, on time of stationary contact, on slip history, and on vibrations are reviewed in [8]. After a possibly appropriate friction law has been chosen, there still remain many issues about the mechanics of slip.

A particularly simple set of mechanics problems relate to the slip of a single rigid body that has simple Coulomb-Amonton-DaVinci friction. A subtle but difficult aspect of these problems follows from the rigid body being finite in spatial extent so that many (or a continuum of) points of contact, each with different slip velocities and friction forces, all contribute to the net friction force and torque. In the context of robotic motion planning, rigid body motion with friction has been studied in [1,6,9]. These authors have looked at the quasi static motion of an object subject to indeterminate contact pressures (normal forces). The dynamics of slip for some special rigid bodies has been discussed in [5,12]. In order to better understand the motion of a rigid body in frictional contact we feel it is useful to pause, and just study the nature of the sum of the friction contact forces on a given body. Certain aspects of the dynamics (and quasi statics) of slip can be inferred from the description of the net friction that results.

Specifically we consider the sum of the frictional forces and moments for the following restricted problem:

1. A rigid body slides on a planar surface. Only the interactions of this body with the surface are studied and not the interactions with any other pushing or restraining objects.
2. The contact normal force (or pressure) distribution is known a priori.
3. At each point of contact, the friction force depends only on the (known) contact force and on the direction of slip and not, for example, on the magnitude of the slip rate, the slip displacement or the slip history.
4. At each point of contact the dependence of the friction force on direction is consistent with a maximum work inequality (to be discussed). The maximum work inequality generalizes Coulomb friction to include, for example, anisotropic contacts mediated by wheels.

Assuming 1-4 above, the friction of a rigid body is also fully characterized by a limit surface exactly of the type used in classical plasticity, e.g. [10]. With the additional restriction of isotropic friction at every point, friction for a rigid body is also fully characterized by the moment function of Zhukovskii [13]. Our goal here is to introduce these geometric descriptions of the net frictional force and torque on a rigid body. Some implications of the descriptions will be mentioned briefly.

The plan of the paper is to first describe the motion of, and the load on a planar rigid body. Then the nature of friction at a single point is discussed. The maximum work inequality is introduced with the resulting concept of a limit curve for a point of contact. The limit curves for Coulomb friction (a circle) and for an ideal wheel (a straight line) are

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given as examples. The maximum work inequality for the overall body is then derived and the resulting concept of a limit surface is introduced and illustrated with an example (a body with two points of support). The moment function is then presented. We conclude with mention of some facts and results related to these descriptions and the dynamics of motion.

The discussion of most of the topics here is presented in an expanded form and with more examples and references in [2,3,4].

Motion of a Rigid Slider

Since we shall be using rate independent friction laws, we only need the direction of the velocity $\mathbf{v}$ at each point of frictional contact to calculate the friction force. These are fully determined by the overall instantaneous motion direction for the planar slider of figure 1 which can be defined either by:

1. The location C about which the body's motion is instantaneously a pure rotation (specified as clockwise or counter-clockwise). The point C is the *center of rotation* (COR) and is on the circle at infinity for pure translations. Or
2. The unit motion vector $\mathbf{q}$, or 'versor' in the language of Zmitrowicz [14], defined as $\mathbf{q} = [q_x, q_y, q_\omega] = \mathbf{Q}/|\mathbf{Q}|$ where $\mathbf{Q} = [V_x, V_y, \omega]$ has components which are the translation velocity and angular velocity of the slider as referenced to a point 'O' on the slider.

(The apparent dimensional inconsistency in the definition of $\mathbf{q}$ above and in the definition of $\mathbf{P}$ below, may be removed by normalizing all length quantities with some characteristic length. Using the radius of gyration of the slider as a length scale aids the geometric interpretation of dynamics calculations.)

The Net Frictional Load

The net frictional load is defined as $\mathbf{P} = [F_x, F_y, M]$ where F_x and F_y are the net force that the planar slider exerts on the support surface and M is the net moment about a vertical axis passing through 'O'. $\mathbf{P}$ is the negative of that which would represent the frictional load on a free body diagram of the object. When inertia is negligible, $\mathbf{P}$ is the total external load which need be applied to the slider to overcome the frictional resistance.

The components of $\mathbf{P}$ can be expressed by integrals over the entire contact region A, of the frictional traction (stress) $\mathbf{f} = [f_{ax}, f_{ay}]$ that the slider causes on the support plane at each contact point with position coordinates $[r_{ax}, r_{ay}]$:

$$F_x = \int_A f_{ax}\, dA, \qquad F_y = \int_A f_{ax}\, dA$$

$$M = \int_A (r_{ax} f_{ay} - r_{ay} f_{ax})\, dA. \qquad (1)$$

The moment of the frictional forces about a vertical axis passing through the center of rotation C is given by:

$$M_c = \int_A ((r_{ax} - r_{cx}) f_{ay} - (r_{ay} - r_{cy}) f_{ax})\, dA. \qquad (2)$$

If point supports are involved, the integrands in (1) and (2) above contain delta functions or, equivalently, are replaced by sums. The sums (1) and (2) and their relation to the motion $\mathbf{q}$ (or C) are the central subject of this presentation.

Isotropic Friction

The simplest friction law for which this work applies is isotropic friction: during slip the friction force (or stress) at a point is in the direction of motion and its magnitude $|\mathbf{f}|$ is a constant independent of the direction of motion. During stick the magnitude of the friction force is less than or equal to this constant. We need not consider the dependence of the friction on the normal force (or stress) since the normal force is assumed known a priori.

An equivalent, and somewhat awkward at first sight, description of this same friction law is the following pair of statements:

1. There is a circle centered at the origin of $[f_x, f_y]$ space. We call this circle the limit curve (LC) for isotropic friction at a given point of contact.
2. The maximum work inequality is always satisfied:

$$(\mathbf{f} - \mathbf{f}^*) \cdot \mathbf{v} \geq \mathbf{0} \qquad (3)$$

where $\mathbf{f}$ and $\mathbf{v}$ are the friction force and the relative slip velocity at a point of contact, $\mathbf{f}$ and $\mathbf{f}^*$ are on or inside the limit curve, and $\mathbf{f}^*$ is otherwise arbitrary.

The situation is shown in figure 2(b) for contact at a point 'a' on the sliding body. The strength of the maximum work inequality (as well as its name) comes from $\mathbf{f}^*$ being arbitrary. For a given $\mathbf{v}$ the corresponding $\mathbf{f}$ is that $\mathbf{f}$, on or inside the limit curve, which maximizes the 'work' (actually power in this formulation) over all possible $\mathbf{f}^*$ on or inside the limit curve (since $\mathbf{f} \cdot \mathbf{v} \geq \mathbf{f}^* \cdot \mathbf{v}$ for all $\mathbf{f}^*$).

Anisotropic Friction

The tool which we use in the construction of the limit surface is the maximum work inequality at each point; the isotropy of friction is not essential. So it is safe, and perhaps useful for some applications, to generalize the friction laws that are allowed to include any law that is described by the statements:

1. A closed curve in force space is specified.
2. The maximum work inequality (eq. 3) is satisfied by all $\mathbf{f}$ and $\mathbf{v}$.

(Much of the reasoning we present based on these statements is borrowed directly from classical plasticity theory, e.g. [10].) The statements 1 and 2 above are only consistent if the specified closed curve is convex and encloses the origin. A possible (if artificial looking) limit curve is shown in figure 3 where the existence of a flat region and a vertex should be noted.

Consequences of the limit curve description are: 1) the normality of $\mathbf{v}$ to the limit curve in places where the limit curve has a well defined normal, thus the friction law is said to 'satisfy normality,' 2) a non-uniqueness in the slip versor $\mathbf{v}/|\mathbf{v}|$ for given friction force at a vertex on the limit curve (all normals to an imagined rounded vertex are possible), and 3) a non-uniqueness in the friction force for given slip velocity $\mathbf{v}$ which is normal to a flat region on the limit curve (all forces on the flat region are possible). These properties are worth noting even if one's ultimate interest is isotropic friction since their analogues on the limit surface for the slider occur even when the limit curve at every point of contact is a circle.

Wheels as Anisotropic Friction

There is some argument about the applicability of normality principles to anisotropic friction in general. However, contact mediated by wheels provides a useful and consistent example [7]. One can imagine an object, say a car or a cart, that one may wish to model using wheeled contact friction. Figure 4 shows a microscopic and massless wheel that has a perfect rigid bearing attached to the rigid body and ordinary isotropic friction at its contact with the ground. This limit curve is highly degenerate in that it encloses no area and has nothing but flat regions and vertices. During rolling the side force is indeterminate (anything on the limit curve), and during side slip any velocity is possible.

Other examples of anisotropic friction can be constructed using a rusty wheel, a wheel with a ratchet, a rust wheel with a ratchet, a wheel with a continuum of ratcheted wheels at its perimeter, etc. On the other hand, one can also invent micro-machines to mediate the contact between the slider and the support plane that violate normality. A microscopic, rusty, castor wheel that is mounted in an appropriately crooked manner violates the maximum work inequality when described macroscopically [2,3,4].

Maximum Work and the Limit Surface

It turns out that the formalism above used to describe friction at a point carries over directly to the rigid body as a whole. The three dimensional friction load $\mathbf{P}$ and the motion versor $\mathbf{q}$ satisfy an inequality similar to the maximum work inequality, with respect to a closed convex limit surface (LS). We term this inequality the load motion inequality. The LS is fully determined by the limit curves of its contact points. In other words, for a given slider (known friction laws at all contact points) the sums in equations (1) evaluated for all possible motions are fully described by a single closed convex surface.

The formal construction of the overall limit surface in load space from the limit curves follows from (3) applied to every point with the principle of virtual work (PVW). The principle of virtual work (actually power) for our rigid body may be expressed as:

$$\mathbf{P}\cdot\mathbf{Q} = \sum \mathbf{f}\cdot\mathbf{v} \quad (4)$$

where $\mathbf{P}$ and $\mathbf{f}$ are any load and force distribution that are related by the 'equilibrium' sums (1) (whether or not the $\mathbf{f}$s are properly associated with any friction laws). $\mathbf{Q} = [V_x, V_y, \omega]$ and $\mathbf{v}$ (slip velocity at position $\mathbf{r}$) are any motion vector and velocity distribution related by the rigid body 'compatibility' equation:

$$\mathbf{v} = [v_x, v_y] = \mathbf{V} - \omega\times\mathbf{r} = [V_x - \omega r_y,\ V_y + \omega r_x]$$

(The cross product should be interpreted as scalar or vector valued, depending on context.) Assume that $\mathbf{f}$ and $\mathbf{f}^*$ are two frictional force distributions that do not violate the slip condition anywhere ($\mathbf{f}$ and $\mathbf{f}^*$ are on or inside their limit curves) and they correspond through (1) respectively to $\mathbf{P}$ and $\mathbf{P}^*$. Further assume that at every point $\mathbf{f}$ is also consistent with the friction law for the velocity field for the motion $\mathbf{Q}$. Applying the principle of virtual work (4) to each of the two cases ($\mathbf{P}\cdot\mathbf{Q} = \sum\mathbf{f}\cdot\mathbf{v}$ and $\mathbf{P}^*\cdot\mathbf{Q} = \sum\mathbf{f}^*\cdot\mathbf{v}$), subtracting, and then applying the inequality (3) to each term, gives:

$$(\mathbf{P} - \mathbf{P}^*)\cdot\mathbf{q} \geq \mathbf{0} \quad (5)$$

So $\mathbf{P}$ is the load vector at O during slip associated with a motion vector $\mathbf{q} = \mathbf{Q}/|\mathbf{Q}|$, and $\mathbf{P}^*$ is any other load vector at O that results from summing possible $\mathbf{f}^*$s (where 'possible' refers to that which is on or inside the limit curves with no attention paid to the velocities at the contact points).

The set of all possible $\mathbf{P}^*$ is the Minkowski sum of the contributions from each of the contact points (once the limit curves for each point of contact from two dimensional force space are replaced by appropriate limit surfaces in three dimensional load space). The boundary of this set is a closed convex surface in load space that encloses the origin. This surface, the limit surface for the object, with the inequality (5) fully describes the relation between friction loads $\mathbf{P}$ and motions $\mathbf{q}$. In short, the net friction on a rigid planar slider, subject to the assumptions named previously, is fully characterized by:

1. Specification of a closed convex surface in load space that encloses the origin. This surface may be regarded as a macroscopic description of the slider, *or* a description constructed from the microscopic contact distribution. And,
2. The load motion inequality (5), $(\mathbf{P} - \mathbf{P}^*)\cdot\mathbf{q} \geq \mathbf{0}$

As with the limit curves, 1) $\mathbf{q}$ is normal to the limit surface where it is smooth, 2) there is a non-uniqueness in the slip versor $\mathbf{q}$ for given friction load at a vertex on the limit surface (all normals to an imagined rounded vertex are possible), and 3) there is a non-uniqueness in the friction load for given motion $\mathbf{q}$ which is normal to a flat region on the limit surface (all loads on the flat region are possible).

Example Limit Surface

Figure 5 shows a symmetric bar supported at its ends by two identical points of isotropic contact, the half length of the bar is 1. The limit surface for this object is shown in figure 6(a). It is the boundary of the volume swept by the convolution of two ellipses, each ellipse being the limit surface for a point of support of the bar. The intersection of the surface with the $[F_x, F_y]$ plane is a circle, as is always the case for bodies whose contact points are governed by isotropic friction. The intersection of the limit surface with the $[M, F_y]$ plane is a circle and the intersection with the $[M, F_x]$ plane is a square as shown in figure 6(b).

The limit surface has 4 vertices where it intersects the F_x and the M axes. It has 4 flat regions showing as elliptical facets whose normals are on the $[M, F_x]$ plane. The end caps at and near the intersections of the LS with the F_y axis are smooth with well defined normals. The vertices on the limit surface correspond to non-uniqueness of the slip motion when either a pure moment or a pure force in the x direction is applied. For example, if a force is applied in the x direction the object may translate or translate with a small amount of rotation and still have the same net friction force. For an object that is supported everywhere with isotropic friction, vertices appear on the limit surface if and only if all support points lie on one straight line.

On this limit surface, a flat elliptical facet corresponds to a whole set of friction loads all of which correspond to rotation about one of the support points. When the bar rotates about a support point that point is not sliding and its friction force can be anything inside its friction circle, hence the non-uniqueness. For an object that is supported by isotropic friction at all points, the limit surface has flat facets if and only if there exist points that carry finite loads. In fact each point of support leads to two parallel flat facets on the limit surface of the slider. An object that has finite contact stress (or even finite force per unit length on a curve of contact) at all points has no flat facets.

When sliders with more general friction laws (such as that describing contact mediated by wheels) are considered, vertices can appear on the limit surface even if all the support is not on a line, and facets can appear even if there are no point supports.

Other symmetries and properties of limit surfaces provide some insight about the relation between forces and the sense of rotation of the object, tendency to rotate about points of support, etc.

Moment Function for Isotropic Friction

We have discovered, and since learned of Zhukovskii's [13] previous discovery of, another simple geometric representation of the friction load for the special case of isotropic friction. For rotation about a point C with position $[x_c, y_c]$ the friction load M_c can be calculated by the sum in equation (2). The value of M_c as a function of $[x_c, y_c]$ is the moment function. It is single valued and well defined since the non-uniqueness in the sums (1) is in the net force $[F_x, F_y]$ and is due to the force of the non-sliding center of rotation. Since M_c is calculated relative to this point, the undetermined friction force at that point has no net moment.

The moment function can be constructed from the sum (2). This sum can be visualized as follows. For every point of support the moment function is a cone that opens upwards with vertex at the point of support on the $[x_c, y_c]$ plane. The moment function is the sum of these cones over all points of contact.

A straightforward calculation shows that the moment function contains all information about the slider's contact. In particular, differentiation of the equations (1) with isotropic friction shows that,

$$F_x = \frac{\partial M_c}{\partial y_c}, \quad F_y = -\frac{\partial M_c}{\partial x_c}, \quad M = M_c - \mathbf{r}_c \times \mathbf{F} \qquad (6)$$

where $\mathbf{r_c}$ is the position of the center of rotation relative to the reference point O and the cross product is scalar valued. The moment function for the bar with two points of support is shown in figure 7. Its gradient is not well defined for the two corners at the bottom. Thus the differentiation above is not sensible and the friction load is not well defined. This corresponds to the COR on a support point, the vertices of the frictional cones of the individual limit cones of the points of contact. The non-uniqueness in $[F_x, F_y]$ at these points is the same as that represented by the elliptical facets on the limit surface.

The moment function has a constant gradient value and constant M_c for all points on the y_c axis between the support points. So that set of CORs corresponds to a set of motions all with the same frictional load. Similarly, on the y_c axis outside of the bar $M_c = \mathbf{r}_c \times \mathbf{F}$ and again a set of CORs is found with a fixed load. These straight lines on the moment function correspond to vertices on the limit surface.

Dynamics Of Free Slip

We have used the limit surface to generalize some results in [5,12] on the dynamics of slip. In particular, any limit surface has at least two 'eigen-directions', directions for which $\mathbf{P}$ is parallel to $\mathbf{q}$, the normal to the surface. When an object comes to rest at the end of free sliding (sliding purely under the action of the friction forces at the frictional interface), the final motion is always (for all initial conditions) in one of these eigen-directions. Some of these eigen-directions are unstable in that they are final motions only for certain points on the limit surface as initial conditions. Others are stable in that they occur as final motions for less restricted initial conditions. The stability of an eigen-direction can be determined from the relation between the principal radii of curvature and the radius of the limit surface at that point. Eigen-directions on flat facets are always stable, thus the final motion of a sliding object is often a rotation about a point of support. Further results on the stability of slip directions have been obtained as applicable to, say, a skidding car or cart.

For dimensional consistency, as well as sensible dynamics, the limit surface uses moments that are scaled by the radius of gyration. Thus the shape of the limit surface depends in part on the distribution of mass. For axisymmetric rigid bodies the final motion is always a pure rotation if the mass is sufficiently far from the center of the object (unless the initial condition is pure translation). The final motion is always pure translation if the mass is sufficiently close to the center of the object (unless the initial condition is pure rotation). These results are discussed at some length in [2,3,4].

Conclusions

The limit surface and moment function provide a geometric description of the overall relation between frictional load and motion of a sliding rigid body. They can be used to visualize or explain many features of rigid body slip of the type demonstrated in [6]. Some interesting results about the dynamics of free slip have also been obtained.

Acknowledgments

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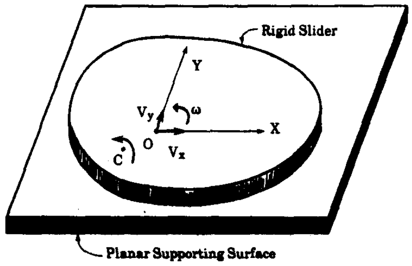


Figure 1. Rigid Slider on a planar surface.

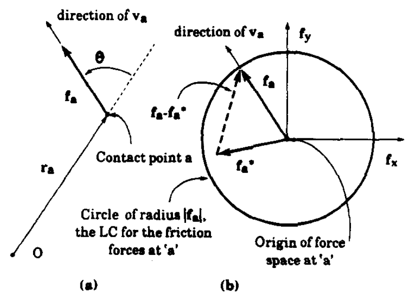


Figure 2. Coulomb friction at a point with position $\mathbf{r_a}$. The friction force is $\mathbf{f_a}$ the slip velocity $\mathbf{v_a}$. $\mathbf{f_a^*}$ is any friction force on or inside the limit curve, which is a circle for isotropic coulomb friction.

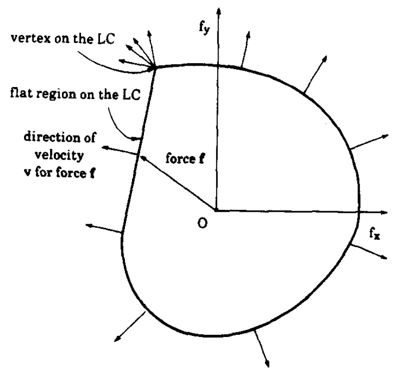


Figure 3. Generic limit curve LC. Friction forces that can occur during sliding for an imagined anisotropic material. The curve is convex and encloses the origin. Where it is smooth velocity is normal to the curve. Where it is flat there is more than one friction force for a given velocity. Where it is kinked there is more than one velocity for a given friction force.

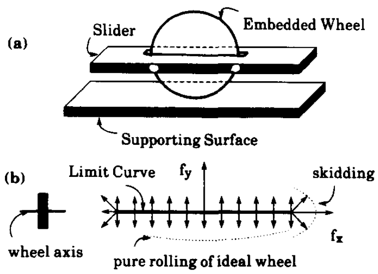


Figure 4. a) Microscopic wheel embedded in the slider. It is massless, has a perfect bearing and makes ordinary frictional contact with the ground. The set of forces it can transmit to the ground, along with the motion of the slider to which they correspond is shown by b) the limit curve for contact mediated by an ideal wheel.

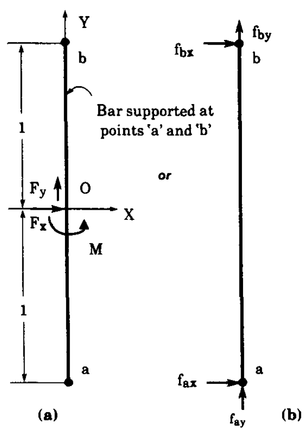
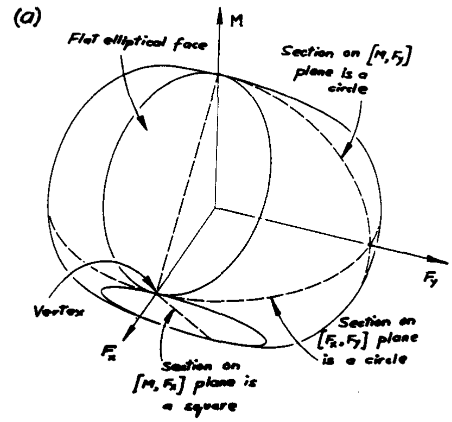


Figure 5. A sample rigid object, a bar supported at its ends. The net frictional load is shown in (a), the contact frictional forces in (b).



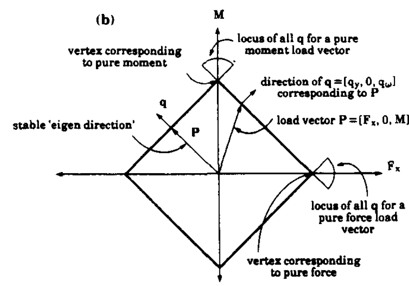


Figure 6. a) The limit surface for the bar of figure 5. The set of all possible frictional loads during slip. b) the section of the limit surface of (6a) on the $[F_x, M]$ plane. Various non-uniquenesses are illustrated with respect to the flat sides and the sharp corners.

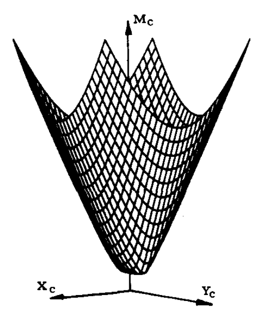


Figure 7. The moment function for the bar of figure (5). The moment function is the sum of two cones each centered at a contact point.